arXiv:1012.4086v2 [math.AG] 27 Jan 2015

Exceptional collections on toric Fano threefolds and birational geometry

Hokuto Uehara

Abstract

Bondal's conjecture states that the Frobenius push-forward of the structure sheaf $\mathcal{O}_X$ generates the derived category $D^b(X)$ for smooth projective toric varieties X.

Bernardi and Tirabassi exhibit a full strong exceptional collection consisting of line bundles on smooth toric Fano 3-folds assuming Bondal's conjecture. In this article, we prove Bondal's conjecture for smooth toric Fano 3-folds and improve upon their result using birational geometry.

[1] [2]

1 Introduction

A full strong exceptional collection of a triangulated category can be thought of as the categorical analogue of a finite orthonormal basis of a vector space. For the derived category $D^b(X)$ of coherent sheaves on a smooth projective variety X, such a collection rarely exists, but if it exists, the derived category $D^b(X)$ is equivalent to the derived category $D^b(\mathrm{mod}\, A)$ of the category $\mathrm{mod}\, A$ of finitely generated modules over a finite dimensional algebra A.

For any smooth toric DM orbifold X, Kawamata shows that there is a full, but not necessarily strong, exceptional collection on X [18]. Furthermore full strong exceptional collections on toric varieties are studied by many people (cf. [3, 8, 9, 10, 11, 15, 20, 17]).

We can define an endomorphism F_m ($m \in \mathbb{Z}_{>0}$) called the *Frobenius map* on any toric variety over a field of any characteristic (some people also call it a *multiplication map*). It is also known that for smooth complete toric varieties X, $F_{m*}\mathcal{O}_X$ splits into line bundles and Thomsen [24] finds an algorithm to compute the set of all direct summands of it. We denote the set by $\mathfrak{D}_X$ for a sufficiently divisible integer m. On the other hand, Bondal's conjecture predicts that the set $\mathfrak{D}_X$ classically generates the derived

[1]Keywords; Derived categories, toric Fano threefolds, birational geometry, exceptional collection

[2]Mathematics Subject Classification 2000: 14E30, 14J30, 14J45, 14M25, 14F05

category $D^b(X)$. So sometimes, for instance in the case $|\mathfrak{D}_X| = \operatorname{rank} K(X)$, it becomes a candidate of a full strong exceptional collection consisting of line bundles on smooth projective toric varieties X.

Bernardi and Tirabassi exhibit such collections on all eighteen smooth toric Fano 3-folds by using Frobenius maps [3]. Using birational geometry, we obtain a stronger result. Precisely we show the following.

Theorem 1.1. *For sixteen smooth toric Fano* 3*-folds X over $\mathbb{C}$, the set $\mathfrak{D}_X$ becomes a full strong exceptional collection (after choosing an appropriate order). For the remaining two cases,* (4) *and* (11) *in Theorem* 3.1*, we present a proper subset of $\mathfrak{D}_X$ which becomes a full strong exceptional collection.*

Note that this theorem implies that Bondal's conjecture is true for smooth toric Fano 3-folds. The strategy to prove Theorem 1.1 is as follows;

Step 1. Let $f\colon X \to Y$ be an extremal birational contractions between smooth toric Fano 3-folds. Assume that $\mathfrak{D}_X$ forms a full strong exceptional collection. Then so is $\mathfrak{D}_Y$. This is done by Lemmas 5.1 and 6.4.

Step 2. By Step 1, it is enough to show that $\mathfrak{D}_X$ forms a full strong exceptional collection only for (birationally) maximal Fano 3-folds X, namely in (11), (17) and (18) in Theorem 3.1. Unfortunately, in the case (11), $\mathfrak{D}_X$ does not form a strong exceptional collection. Instead we can find a subset $\mathfrak{D}_{nef}$ of $\mathfrak{D}_X$ which becomes a full strong exceptional collection. Then, as in Step 1, an inductive argument works in the case X in (11).

Step 3. To check the strongness of the chosen set in Step 2, it is enough to check the dual of line bundles in the set are nef (Lemma 3.8). This is easily done by observing Figure 5. To check the fullness in Step 2, we prove Bondal's conjecture in our situation by rather tedious, but elementary calculation. This step is done in §4 and §5.

In [3], Bernardi and Tirabassi check a similar statement to Theorem 1.1 only in the cases (9), (11), (14), (15) and (16) separately, since the existence of a full strong exceptional collection, not necessarily coming from $\mathfrak{D}_X$, was already known in the remaining cases. In their proof of the strongness, a rather long and explicit calculation is presented, using the result in [7]. Moreover they deduce fullness for their collections only after assuming the conjecture of Bondal (see Remark 3.7).

Dubrobin conjectures that for a smooth projective variety X, the quantum cohomology of X is semi-simple if and only if X is a smooth Fano variety with a full exceptional collection. Although his conjecture turns out to be wrong, it is still believed that there is a relationship between the existence of full exceptional collections on X and its quantum cohomology (cf. [2].)

Furthermore several people conjectured that every smooth toric Fano variety has a full strong exceptional collection consisting of line bundles [7, 8]. But recently Efimov provides a counterexample to this conjecture [12]. At least in the 3-dimensional case, the conjecture is true by Theorem 1.1.

It should be pointed out that there is a smooth projective toric (not Fano) surface which does not possess any full strong exceptional collections consisting of line bundles ([14, 15]), and it is also worthwhile to mention that there is a smooth toric Fano variety X such that we cannot choose full strong exceptional collections from the set $\mathfrak{D}_X$ [20].

Because our exceptional collections consist of line bundles, the corresponding quivers, Gram matrices etc., should be rather easy to be computed. Furthermore our collection exhibits nice properties, as it behaves well as in Step 1 above.

The structure of this paper is as follows: In §2, we give some basic definitions on the derived categories of coherent sheaves. In §3, we explain several notion on toric varieties and cite some useful results for smooth toric Fano 3-folds. We also explain how to determine the set $\mathfrak{D}_X$, following Thomsen. In §4, we actually determine it for several toric varieties. In §5, we prove Bondal's conjecture for maximal smooth toric Fano 3-folds. In §6, we accomplish Step 1 above and give the proof of Theorem 1.1. In Theorem 6.3 we also obtain a similar result in the surface case to Theorem 1.1.

Notation and conventions For a smooth variety X, we denote the bounded derived category of coherent sheaves on X by $D^b(X)$. *T-invariant* is an abbreviation of *torus invariant.* For objects $\mathcal{E}, \mathcal{F} \in D^b(X)$, we define

$$\operatorname{Hom}^i_X(\mathcal{E}, \mathcal{F}) := \operatorname{Hom}_{D^b(X)}(\mathcal{E}, \mathcal{F}[i]).$$

We work over $\mathbb{C}$ for simplicity.

We denote by $M(l, m)$ the space of $l \times m$ matrices defined over $\mathbb{Z}$, and ${}^tA$ is the transpose of a matrix $A \in M(l, m)$

2 Generators of derived categories

In this section, we give several basic definitions on triangulated categories and derived categories of coherent sheaves.

Definition 2.1. Let $\mathcal{S} = \{\mathcal{S}_i\}$ be a set of objects in a triangulated category $\mathcal{D}$.

(i) We denote by $\langle \mathcal{S} \rangle$ the smallest triangulated subcategory of $\mathcal{D}$ containing all $\mathcal{S}_i$, closed under isomorphisms and direct summands. For a triangulated subcategory $\mathcal{C}$ of $\mathcal{D}$, we denote by $\mathcal{C}^\perp$ the full triangulated subcategory of $\mathcal{D}$ whose objects $\mathcal{F}$ satisfy the property $\operatorname{Hom}_{\mathcal{D}}(C, \mathcal{F}) = 0$ for all $C \in \mathcal{C}$.

(ii) We say that $\mathcal{S}$ *classically generates* $\mathcal{D}$ if $\langle \mathcal{S} \rangle = \mathcal{D}$. We also call $\mathcal{S}$ a *classical generator* of $\mathcal{D}$.

(iii) We say that $\mathcal{S}$ *generates* $\mathcal{D}$ if $\langle \mathcal{S} \rangle^{\perp} = 0$. We also call $\mathcal{S}$ a *generator* of $\mathcal{D}$.

Let X be a smooth complete variety over $\mathbb{C}$.

Definition 2.2. (i) An object $\mathcal{E} \in D^b(X)$ is called *exceptional* if it satisfies

$$\mathrm{Hom}_X^i(\mathcal{E}, \mathcal{E}) = \begin{cases} \mathbb{C} & i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

(ii) An ordered set $(\mathcal{E}_1, \dots, \mathcal{E}_n)$ of exceptional objects is called an *exceptional collection* if the following condition holds;

$$\mathrm{Hom}_X^i(\mathcal{E}_k, \mathcal{E}_j) = 0$$

for all $k > j$ and all i. When we say that a finite set $\mathcal{S}$ of objects is an exceptional collection, it means that $\mathcal{S}$ forms an exceptional collection after choosing an appropriate order.

(iii) An exceptional collection $(\mathcal{E}_1, \dots, \mathcal{E}_n)$ is called *strong* if

$$\mathrm{Hom}_X^i(\mathcal{E}_k, \mathcal{E}_j) = 0$$

for all k, j and $i \neq 0$.

(iv) An exceptional collection $(\mathcal{E}_1, \dots, \mathcal{E}_n)$ is called *full* if

$$\langle \mathcal{E}_1, \dots, \mathcal{E}_n \rangle = D^b(X).$$

Remark 2.3. If X has a full exceptional collection consisting of n exceptional objects, the rank of its K-group $K(X)$ is n ([5]). Furthermore it is known that the rank of K-group is the number of the maximal cones in the fan for smooth projective toric varieties (cf. [13, Theorem in §5.2.]). For 3-dimensional smooth projective toric varieties X, we can see $\operatorname{rank} K(X) = 2\rho(X) + 2$.

3 Toric varieties

Throughout this section, we use the following notation. Let $N = \mathbb{Z}^n$ be a lattice of rank n and M its dual. A fan Δ in $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ consists of a finite number of rational strongly convex polyhedral cones in $N_{\mathbb{R}}$, and it determines a toric variety $X = X(\Delta)$. We define $\mathcal{V}(\Delta)$ to be the set of primitive generators of 1-dimensional cones in Δ.

For a cone σ in Δ, put $R_\sigma = \mathbb{C}\big[\chi^{\boldsymbol{u}} \bigm| \boldsymbol{u} \in M \cap \sigma^\vee\big]$, where $\big\{\chi^{\boldsymbol{u}} \bigm| \boldsymbol{u} \in M \cap \sigma^\vee\big\}$ is a basis of $\mathbb{C}$-vector space R_σ. We can define an obvious multiplication on R as usual (cf. [13, page 15]). Then the affine toric variety U_σ corresponds to σ is just $\operatorname{Spec} R_\sigma$, and the rational function field of the toric varieties X is just $\mathbb{C}\big[\chi^{\boldsymbol{u}} \bigm| \boldsymbol{u} \in M\big]$.

3.1 Double $\mathbb{Z}$-weight

According to [21], we introduce the notion of doubly $\mathbb{Z}$-weighted triangulations of a 2-sphere. For simplicity, we restrict to the case $n = 3$, namely $N = \mathbb{Z}^3$ below. We can obviously identify the set of half lines starting from the origin $\mathbf{0}$ of $N_\mathbb{R}$ with

$$S^2 := (N_\mathbb{R}\backslash\{\mathbf{0}\})/\mathbb{R}_{>0}.$$

Let

$$\pi \colon N_\mathbb{R}\backslash\{\mathbf{0}\} \to S^2$$

be the projection. We call $\pi(v)$ a *rational point* of S^2 corresponding to a primitive element $v \in N$, and v is called the *N-weight* of the rational point $\pi(v)$. For a cone $\sigma = \mathbb{R}_{\geq 0}\boldsymbol{v}_1 + \cdots + \mathbb{R}_{\geq 0}\boldsymbol{v}_s \in \Delta$ ($\boldsymbol{v}_i \in \mathcal{V}(\Delta)$), $\pi(\sigma\backslash\{\mathbf{0}\})$ is a convex spherical cell in S^2 with rational points $\pi(\boldsymbol{v}_1), \ldots, \pi(\boldsymbol{v}_s)$ as vertices. Thus for a fan Δ, we get a convex spherical cell decomposition

$$\big\{\pi(\sigma\backslash\{\mathbf{0}\} \bigm| \sigma \in \Delta\big\}$$

of $\pi(|\Delta|\backslash\{\mathbf{0}\})$.

Suppose that a fan Δ is complete and non-singular, which is equivalent to the condition that the corresponding toric variety $X = X(\Delta)$ is proper and smooth. Then we get a simplicial cell decomposition of S^2. Moreover, for each 3-dimensional cone $\sigma \in \Delta$, the corresponding spherical 2-simplex $\pi(\sigma\backslash\{\mathbf{0}\})$ has vertices whose N-weights $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ form a $\mathbb{Z}$-basis of N. For each 2-dimensional cone $\tau \in \Delta$, there are exactly two 3-dimensional cone $\sigma, \sigma' \in \Delta$ such that $\sigma \cap \sigma' = \tau$. In this case, the sets $\{\boldsymbol{v}, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ and $\{\boldsymbol{v}', \boldsymbol{v}_2, \boldsymbol{v}_3\}$ of N-weights for the vertices of $\pi(\sigma\backslash\{\mathbf{0}\})$ and $\pi(\sigma'\backslash\{\mathbf{0}\})$, respectively, are $\mathbb{Z}$-bases of N. Moreover we have

$$\boldsymbol{v} + \boldsymbol{v}' + \alpha_2\boldsymbol{v}_2 + \alpha_3\boldsymbol{v}_3 = \mathbf{0}$$

for $\alpha_j \in \mathbb{Z}$ uniquely determined by τ. For

$$\rho = \mathbb{R}_{\geq 0}\boldsymbol{v}, \rho' = \mathbb{R}_{\geq 0}\boldsymbol{v}', \rho_j = \mathbb{R}_{\geq 0}\boldsymbol{v}_j \in \Delta \quad (j = 2, 3)\ ,$$

let D, D', D_j are the corresponding T-invariant divisors. Then it is known (cf. [21, page 81]) that we have

$$\alpha_j = (D_j \cdot D_2 \cdot D_3). \tag{1}$$

We then endow the edge $\pi(\boldsymbol{v}_2), \pi(\boldsymbol{v}_3)$ with the *double* $\mathbb{Z}$*-weight* α_2, α_3, where we place α_2 (resp. α_3) on the side of the vertex $\pi(\boldsymbol{v}_2)$ (resp. $\pi(\boldsymbol{v}_3)$) as in Figure 1. For simplicity, here and henceforth we always denote a rational point $\pi(\boldsymbol{v})$ by its N-weight $\boldsymbol{v}$ in figures of double $\mathbb{Z}$-weights.

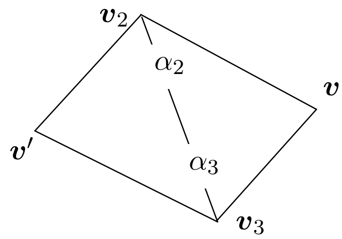


Figure 1: Double $\mathbb{Z}$-weight

We have normal bundle sequences;

$$\begin{aligned} 0 \to &\mathcal{N}_{C/D_2} \to \mathcal{N}_{C/X} \to \mathcal{N}_{D_2/X}|_C \to 0 \\ 0 \to &\mathcal{N}_{C/D_3} \to \mathcal{N}_{C/X} \to \mathcal{N}_{D_3/X}|_C \to 0, \end{aligned}$$

where $C \cong \mathbb{P}^1$ is the T-invariant curve corresponding to the cone τ. Then we know as above $\mathcal{N}_{C/D_j} \cong \mathcal{O}_{\mathbb{P}^1}(-\alpha_j)$ and $\mathcal{N}_{D_j/X}|_C \cong \mathcal{O}_{\mathbb{P}^1}(-\alpha_{j'})$, where $\{j, j'\} = \{2, 3\}$. Combining both sequences, we conclude that they split and so we have

$$\mathcal{N}_{C/X} \cong \mathcal{O}_{\mathbb{P}^1}(-\alpha_2) \oplus \mathcal{O}_{\mathbb{P}^1}(-\alpha_3). \tag{2}$$

We show in Figure 2 the change of double $\mathbb{Z}$-weights under the blowing-up along a T-invariant curve [21, page 90]. The segment attached to a oval corresponds to the curve, and the vertex with a dark gray small circle corresponds to the exceptional divisor. Figure 2 will be used to find the centers and the exceptional divisors of blowing-ups in Figure 4.

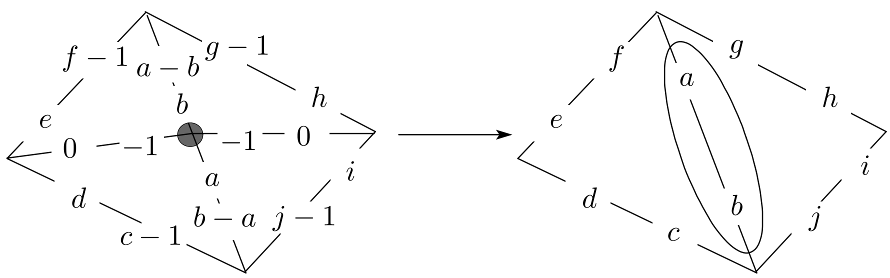


Figure 2: Change of double $\mathbb{Z}$-weights under the blowing-up

3.2 Toric Fano 3-folds

Smooth toric Fano 3-folds are classified as follows.

Theorem 3.1 ([1, 26])**.** *Up to isomorphism, there are* 18 *distinct Fano* 3*-folds. Among them, each of* $(11), (12), (14), (15), (16)$ *and* (18) *below is obtained from one of the others by a finite succession of equivariant blowing-ups.*

(1) $\mathbb{P}^3$.

(2) $\mathbb{P}^2 \times \mathbb{P}^1$.

(3) *The* $\mathbb{P}^1$*-bundle* $\mathbb{P}(\mathcal{O}_Y \oplus \mathcal{O}_Y(1))$ *over* $Y = \mathbb{P}^2$.

(4) *The* $\mathbb{P}^1$*-bundle* $\mathbb{P}(\mathcal{O}_Y \oplus \mathcal{O}_Y(2))$ *over* $Y = \mathbb{P}^2$.

(5) *The* $\mathbb{P}^2$*-bundle* $\mathbb{P}(\mathcal{O}_Y \oplus \mathcal{O}_Y \oplus \mathcal{O}_Y(1))$ *over* $Y = \mathbb{P}^1$.

(6) $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

(7) *The* $\mathbb{P}^1$*-bundle* $\mathbb{P}(\mathcal{O}_Y \oplus \mathcal{O}_Y(f_1 + f_2))$ *over* $Y = \mathbb{P}^1 \times \mathbb{P}^1$*, where* f_1 *and* f_2 *are fibers of the two projections from* Y *to* $\mathbb{P}^1$.

(8) $\mathbb{P}(\mathcal{O}_Y \oplus \mathcal{O}_Y(f_1 - f_2))$ *in the notation of* (7).

(9) $\mathbb{P}^1 \times \Sigma_1$ *for the Hirzeburch surface* Σ_1.

(10) *The* $\mathbb{P}^1$*-bundle* $\mathbb{P}(\mathcal{O}_Y \oplus \mathcal{O}_Y(s + f))$ *over* $Y = \Sigma_1$*, where* f *is a fiber of the* $\mathbb{P}^1$*-bundle on* Σ_1 *and* s *is the minimal section with* $s^2 = -1$.

(13) $\mathbb{P}^1 \times Y_2$*, where* Y_2 *is the toric del Pezzo surface obtained from* $\mathbb{P}^2$ *by the equivariant blowing-up at two of the* T*-invariant points.*

(17) $\mathbb{P}^1 \times Y_3$*, where* Y_3 *is the toric del Pezzo surface obtained from* $\mathbb{P}^2$ *by the equivariant blowing-up at the three* T*-invariant points.*

Their birational relations are described in Figure 3. There are just three maximal Fano 3*-folds,* (11)*,* (17) *and* (18)*, with respect to birational relations.*

The corresponding eighteen doubly $\mathbb{Z}$-weighted triangulations of S^2 are given in Figure 4. A segment attached to an oval corresponds to the T-invariant smooth curve of the center of a blowing-up appearing in Figure 3. The number, like "(5)" in Figure 4(1), near the oval is the number of the Fano 3-fold obtained from the blowing-up.

For instance, the oval in Figure 4(1) means that if we blow up along the curve corresponding to the segment with the oval, we obtain the Fano 3-fold in (5). Of course, in this case, by symmetry we can choose any other segments, or any other T-invariant smooth curves, as the blowing-up center.

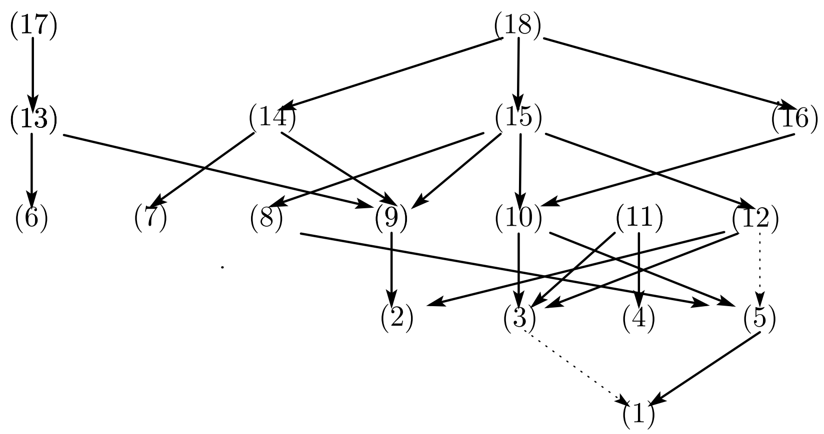


Figure 3: Every arrow means the equivariant blowing-up along a T-invariant curve and every dotted arrow means the equivariant blowing-up along a T-invariant point.

A dark gray small circle at a vertex corresponds to the exceptional T-invariant divisor of a blowing-down appearing in Figure 3. The number, like "(1)" in Figure 4(5), near the small circle is the number of the Fano 3-fold obtained from the blowing-down.

For instance, the small circle in Figure 4(5) means that if we blow down the T-invariant divisor corresponding to the vertex with the small circle, we obtain the Fano 3-fold in (1).

In Figure 4, we do not indicate the point of a blowing-up center, or the exceptional divisor of a blowing-down to a point, since we do not need it afterwards.

3.3 Frobenius push-forward

In this subsection, we explain how to compute the direct summands of Frobenius push-forward of line bundles on smooth complete toric varieties, following Thomsen [24].

Fix a positive integer m and we define a new lattice N' as $N' := \frac{1}{m}N$ and denote its dual by M', We consider the natural inclusion $f_m \colon N \hookrightarrow N'$, which sends a cone in $N_{\mathbb{R}}$ to the cone with the same support on $N'_{\mathbb{R}}$. Thus f_m induces the finite surjective toric morphism $F_m \colon X(\Delta) \to X(\Delta)$ which we call a *Frobenius* (or *multiplication*) *map.* We put

$$\mathcal{V}(\Delta) = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_l\},$$

and D_i to be the T-invariant divisor corresponding to the 1-dimensional cone generated by $\boldsymbol{v}_i$. Henceforth, without otherwise specified, we always assume that Δ is a complete smooth fan, i.e. $X = X(\Delta)$ is a smooth complete toric

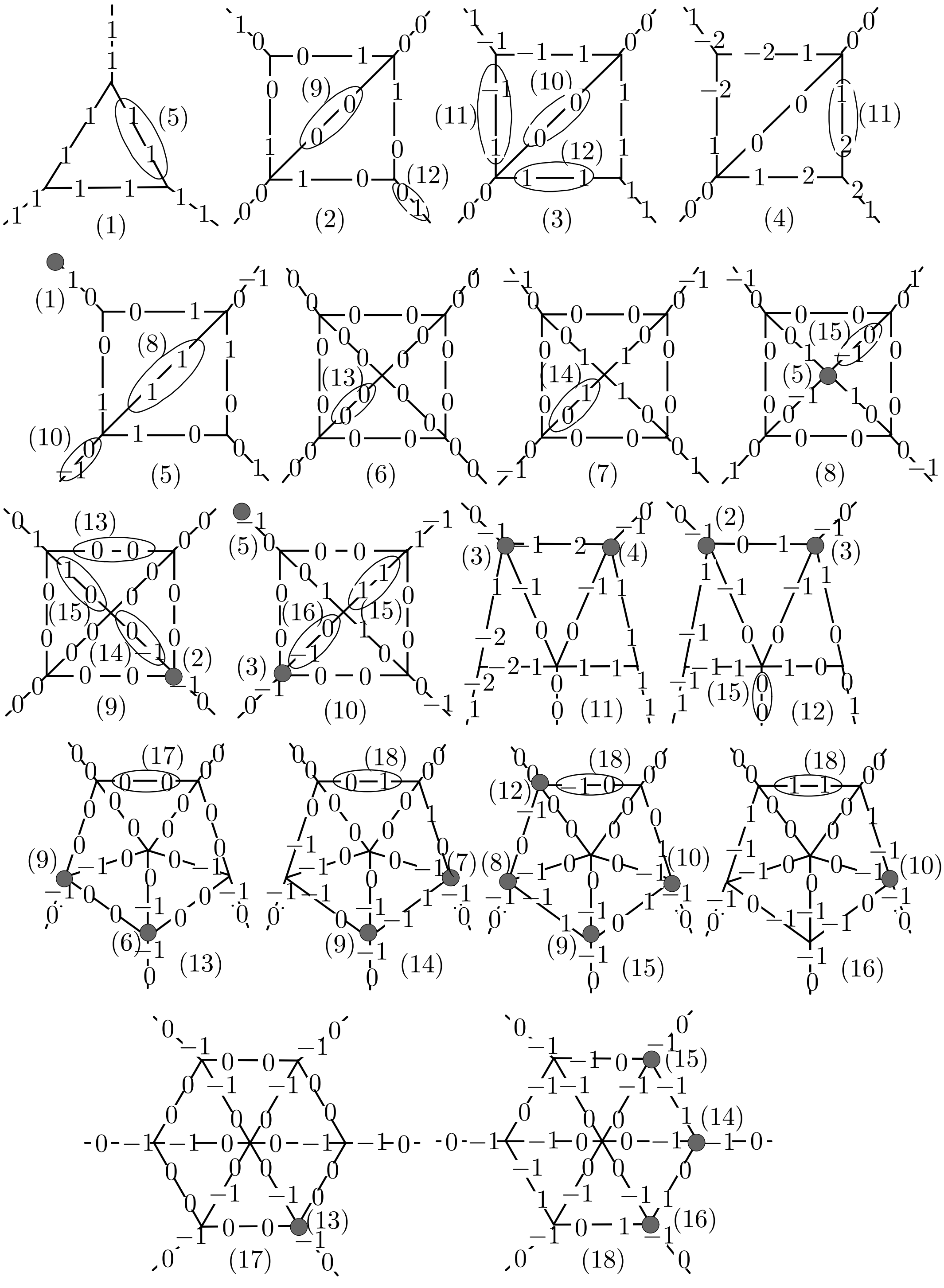


Figure 4: Doubly $\mathbb{Z}$-weighted triangulations of S^2 [21, page 91]

variety. We put

$$A = {}^t(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_l) \in M(l, n).$$

If $D = \sum_{j=1}^{l} b_j D_j$ is a $\mathbb{Q}$-divisor, we define

$$\lceil D \rceil := \sum_{j=1}^{l} \lceil b_i \rceil D_j,$$

where for any real number x, $\lceil x \rceil$ is the integer defined by $x \leq \lceil x \rceil < x + 1$. Similarly, we define

$$\lfloor D \rfloor := \sum_{j=1}^{l} \lfloor b_j \rfloor D_j,$$

where for every x, $\lfloor x \rfloor$ is the integer defined by $x - 1 < \lfloor x \rfloor \leq x$. K_X denotes the canonical divisor $-\sum_{j=1}^{l} D_j$ so that $\omega_X = \mathcal{O}_X(K_X)$.

For a maximal cone

$$\sigma = \langle \boldsymbol{v}_{i_1}, \ldots, \boldsymbol{v}_{i_n} \rangle \in \Delta \quad (i_1 < \cdots < i_n)$$

and a matrix

$$B = {}^t(\boldsymbol{b}_1, \ldots, \boldsymbol{b}_l) \in M(l, m) \quad (l \geq i_n, m \geq 1),$$

we define

$$B_\sigma = {}^t(\boldsymbol{b}_{i_1}, \ldots, \boldsymbol{b}_{i_n}) \in M(n, m).$$

Then note that for a maximal cone σ and the matrix A defined above, A_σ belongs to $GL(n, \mathbb{Z})$, since X is a smooth toric variety.

Put

$$P_m^p = \{ {}^t(u_1, \ldots, u_p) \in \mathbb{Z}^p \bigm| 0 \leqq u_i < m \}$$

for a positive integer p. For $\boldsymbol{u} \in P_m^n$, $\boldsymbol{w} = {}^t(w_1, \ldots, w_l) \in \mathbb{Z}^l$ and a maximal cone $\sigma \in \Delta$, define $\boldsymbol{q}^m(\boldsymbol{u}, \boldsymbol{w}, \sigma) \in \mathbb{Z}^l, \boldsymbol{r}^m(\boldsymbol{u}, \boldsymbol{w}, \sigma) \in P_m^l$ and $q_i^m(\boldsymbol{u}, \boldsymbol{w}, \sigma) \in \mathbb{Z}$ as

$$AA_\sigma^{-1}(\boldsymbol{u} - \boldsymbol{w}_\sigma) + \boldsymbol{w} = m\boldsymbol{q}^m(\boldsymbol{u}, \boldsymbol{w}, \sigma) + \boldsymbol{r}^m(\boldsymbol{u}, \boldsymbol{w}, \sigma) \tag{3}$$

and

$$\boldsymbol{q}^m(\boldsymbol{u}, \boldsymbol{w}, \sigma) = {}^t(q_1^m(\boldsymbol{u}, \boldsymbol{w}, \sigma), \ldots, q_l^m(\boldsymbol{u}, \boldsymbol{w}, \sigma)).$$

Define

$$D_{\boldsymbol{u},\boldsymbol{w},\sigma} (= D^m_{\boldsymbol{u},\boldsymbol{w},\sigma}) := \sum q_i^m(\boldsymbol{u}, \boldsymbol{w}, \sigma) D_i.$$

Remark 3.2. (i) Suppose that

$$\boldsymbol{a} - \boldsymbol{b} = A\boldsymbol{u}$$

for $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{Z}^l$ and $\boldsymbol{u} \in \mathbb{Z}^n$. Then we know that

$$\sum_{i=1}^{l} a_i D_i - \sum_{i=1}^{l} b_i D_i = \operatorname{div} \chi^{\boldsymbol{u}}.$$

In particular the divisors $\sum_{i=1}^{l} a_i D_i$ and $\sum_{i=1}^{l} b_i D_i$ are linearly equivalent.

(ii) We have $\operatorname{div} \chi^{A_\sigma^{-1}\boldsymbol{q}}|_{U_\sigma} = \sum_{i=1}^{n} q_i D_i|_{U_\sigma}$ for any $\boldsymbol{q} = {}^t(q_1, \ldots, q_n) \in \mathbb{Z}^n$.

Example 3.3. Put $R = R_\sigma$ for an n-dimensional non-singular strongly convex rational cone σ in N. For a smooth affine toric variety $U = \operatorname{Spec} R$, the multiplication map F_m induces a $\mathbb{C}$-algebra map

$$F_m^{\#} \colon R \to R \quad \chi^{\boldsymbol{u}} \mapsto \chi^{m\boldsymbol{u}}.$$

When we regard a quotient field K of R as an R-module via the map $F_m^{\#}$, we denote it by $F_{m*}K$. For a sub-R-module L of K, we also define a sub-R-module $F_{m*}L$ of $F_{m*}K$, which is just L as an abelian group.

Then the module $F_{m*}(R\chi^{-A_\sigma^{-1}\boldsymbol{w}})$ for some $\boldsymbol{w} \in \mathbb{Z}^n$ is freely generated by the set

$$\left\{\chi^{A_\sigma^{-1}(\boldsymbol{u}-\boldsymbol{w})} \;\middle|\; \boldsymbol{u} \in P_m^n\right\},$$

namely, we have an isomorphism

$$F_{m*}(R\chi^{-A_\sigma^{-1}\boldsymbol{w}}) \cong \bigoplus_{\boldsymbol{u} \in P_m^n} R\chi^{A_\sigma^{-1}(\boldsymbol{u}-\boldsymbol{w})},$$

since we also have the following description;

$$R = k\left[\chi^{A_\sigma^{-1}\boldsymbol{e}_i} \;\middle|\; i = 1, \ldots, n\right],$$

where $\left\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\right\}$ is the standard basis of M.

The isomorphism on an affine piece in Example 3.3 can be globalized as follows in (ii).

Lemma 3.4. *Fix a vector* $\boldsymbol{w} = {}^t(w_1, \ldots, w_l) \in \mathbb{Z}^l$ *and a maximal cone* $\sigma \in \Delta$.

(i) [24] The vector bundle

$$\bigoplus_{\boldsymbol{u} \in P_m^n} \mathcal{O}_X(D_{\boldsymbol{u},\boldsymbol{w},\sigma})$$

does not depend on the choice of a maximal cone σ.

(ii) [24] We have

$$\bigoplus_{\boldsymbol{u}\in P_m^n} \mathcal{O}_X(D_{\boldsymbol{u},\boldsymbol{w},\sigma}) \cong F_{m*}\mathcal{O}_X(\sum w_i D_i).$$

(iii) For a line bundle $\mathcal{L} \in \operatorname{Pic} X$, we have

$$(F_{m*}\mathcal{L})^\vee \cong F_{m*}(\mathcal{L}^\vee \otimes \omega_X^{1-m}) \cong F_{m*}(\mathcal{L}^\vee \otimes \omega_X) \otimes \omega_X^{-1}.$$

Proof. (iii) The second isomorphism follows from the projection formula. The first one is a direct consequence of the Grothendieck–Verdier duality (cf. [16, page 86]), but we give another proof by the use of the above result. Put $\mathcal{L} = \mathcal{O}_X(\sum w_i D_i)$. We have $\mathcal{L}^\vee \otimes \omega_X^{1-m} = \mathcal{O}_X(\sum (m-1-w_i)D_i)$. Put $\boldsymbol{u}' = (m-1)\,{}^t(1,1,\dots,1)$. Then, for all $\boldsymbol{u} \in P_m^n$, we can see

$$\begin{aligned}
q_i(\boldsymbol{u}'_{\sigma_0} - \boldsymbol{u}, \boldsymbol{u}' - \boldsymbol{w}, \sigma_0) =& \lfloor \frac{{}^t\boldsymbol{v}_i(\boldsymbol{u}'_{\sigma_0} - \boldsymbol{u} - \boldsymbol{u}'_{\sigma_0} + \boldsymbol{w}_{\sigma_0}) - w_i + m - 1}{m} \rfloor \\
=& \lfloor - \frac{{}^t\boldsymbol{v}_i(\boldsymbol{u} - \boldsymbol{w}_{\sigma_0}) + w_i + 1}{m} \rfloor + 1 \\
=& - \lceil \frac{{}^t\boldsymbol{v}_i(\boldsymbol{u} - \boldsymbol{w}_{\sigma_0}) + w_i}{m} + \frac{1}{m} \rceil + 1 \\
=& - q_i(\boldsymbol{u}, \boldsymbol{w}, \sigma_0).
\end{aligned}$$

The last equality holds because, in general, the equality $\lceil \frac{k}{m} + \frac{1}{m} \rceil - \lfloor \frac{k}{m} \rfloor = 1$ is true for $k \in \mathbb{Z}$. This gives the first isomorphism by (iii). $\square$

Below for simplicity, we often identify two isomorphic line bundles. For a T-invariant divisor D and an integer $m > 0$, we define sets of (isomorphism classes of) line bundles;

$$\mathfrak{D}(D)_m := \{\mathcal{L} \in \operatorname{Pic} X \bigm| \mathcal{L} \text{ is a direct summand of } F_{m*}\mathcal{O}_X(D)\ \}$$

and

$$\mathfrak{D}(D) := \cup_{m>0} \mathfrak{D}(D)_m.$$

Convention

(i) We may assume that $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ forms a standard basis of $\mathbb{Z}^n$ and put $\sigma_0 = \langle \boldsymbol{v}_1, \dots, \boldsymbol{v}_n \rangle$. We often omit σ_0 in the notation as $\boldsymbol{q}^m(\boldsymbol{u}, \boldsymbol{w})(:= \boldsymbol{q}^m(\boldsymbol{u}, \boldsymbol{w}, \sigma_0))$, $D_{\boldsymbol{u},\boldsymbol{w}}(:= D_{\boldsymbol{u},\boldsymbol{w},\sigma_0})$ and so on.

(ii) For a zero divisor $D = 0$ or a zero vector $\boldsymbol{w} = \boldsymbol{0}$, we simply denote $\mathfrak{D}(0)$ by $\mathfrak{D}$ (or $\mathfrak{D}_X$ if we need to specify the base variety X) and $\boldsymbol{q}^m(\boldsymbol{u})(= \boldsymbol{q}^m(\boldsymbol{u}, \boldsymbol{0}))$. (In fact, as a consequence of Lemma 3.5(i) and (ii), we have $\mathfrak{D} = \mathfrak{D}(0)_m$ for a sufficiently divisible integer m.)

We take σ in (3) to be σ_0, then (3) becomes the following simpler form

$$A(\boldsymbol{u} - \boldsymbol{w}_{\sigma_0}) + \boldsymbol{w} = m\boldsymbol{q}^m(\boldsymbol{u}, \boldsymbol{w}) + \boldsymbol{r}^m(\boldsymbol{u}, \boldsymbol{w}), \tag{4}$$

and hence we have

$$q_i^m(\boldsymbol{u}, \boldsymbol{w}) = \lfloor \frac{{}^t\boldsymbol{v}_i(\boldsymbol{u} - \boldsymbol{w}_{\sigma_0}) + w_i}{m} \rfloor. \tag{5}$$

Lemma 3.5. *Fix a T-invariant divisor $D = \sum w_i D_i$ and put $\boldsymbol{w} = {}^t(w_1, \dots, w_l)$.*

(i) [24] The set $\mathfrak{D}(D)$ is finite.

(ii) Put $D' := \sum w_i' D_i$, where

$$w_i' := \begin{cases} 0 & \text{for } i \text{ with } w_i \geq 0 \\ -1 & \text{for } i \text{ with } w_i < 0. \end{cases}$$

Take $m > 0$ satisfying $-1 \leq \frac{w_i}{m} < 1$ for any i. Then $\mathcal{O}_X(D') \in \mathfrak{D}(D)_m$. Furthermore we have $\mathfrak{D}(D') \subset \mathfrak{D}(D)_m$ for sufficiently divisible integers $m > 0$.

(iii) We have $\mathfrak{D}(D)_m \subset \mathfrak{D}(lD)_{lm}$ for any $l, m \in \mathbb{Z}_{>0}$.

Proof. (i) Since the set $\left\{ \frac{{}^t\boldsymbol{v}_i\boldsymbol{u}}{m} \;\middle|\; \boldsymbol{u} \in P_m^n, m \in \mathbb{Z}_{>0} \right\}$ is bounded and $\frac{{}^t\boldsymbol{v}_i\boldsymbol{w}_{\sigma_0} - w_i}{m} \to 0$ as $m \to \infty$, the set of integers $\left\{ q_i^m(\boldsymbol{u}, \boldsymbol{w}) \;\middle|\; \boldsymbol{u} \in P_m^n, m \in \mathbb{Z}_{>0} \right\}$ is finite. Consequently, so is the set $\mathfrak{D}(D)$.

(ii) For any $\boldsymbol{w}_{\sigma_0} \in \mathbb{Z}^n$, there is a vector $\boldsymbol{u}' \in \mathbb{Z}^n$ such that $m\boldsymbol{u}' + \boldsymbol{w}_{\sigma_0} \in P_m^n$. Put $\boldsymbol{u} := m\boldsymbol{u}' + \boldsymbol{w}_{\sigma_0}$. Then we can see

$$\begin{aligned} q_i^m(\boldsymbol{u}, \boldsymbol{w}) =& \lfloor \frac{{}^t\boldsymbol{v}_i(\boldsymbol{u} - m\boldsymbol{u}' - \boldsymbol{w}_{\sigma_0}) + w_i}{m} \rfloor + \frac{m\,{}^t\boldsymbol{v}_i\boldsymbol{u}'}{m} \\ =& \lfloor \frac{w_i}{m} \rfloor + {}^t\boldsymbol{v}_i\boldsymbol{u}' = w_i' + {}^t\boldsymbol{v}_i\boldsymbol{u}', \end{aligned}$$

which means the divisor $D_{\boldsymbol{u},\boldsymbol{w}}$ is linearly equivalent to D'. Thus $\mathcal{O}_X(D') \in \mathfrak{D}(D)_m$.

Take an element $\mathcal{L} \in \mathfrak{D}(D')$ and suppose that $\mathcal{L} \in \mathfrak{D}(D)_m$ for some m satisfying $-1 \leq \frac{w_i}{m} < 1$ for any i. Then $\mathcal{L} \in \mathfrak{D}(D)_{km}$ for all $k > 0$, since $F_{km*}\mathcal{O}_X(D) = F_{m*}F_{k*}\mathcal{O}_X(D)$. This gives the last assertion.

(iii) By definition, we have $q_i^m(\boldsymbol{u}, \boldsymbol{w}) = q_i^{lm}(l\boldsymbol{u}, l\boldsymbol{w})$, which implies the conclusion. □

My optimistic conjecture is as follows;

Conjecture 3.6. *Let X be a smooth complete toric variety. Then $F_{m*}\mathcal{O}_X(D)$ is a classical generator of $D^b(X)$ for any T-invariant divisors D and a sufficiently large integer m.*

In order to prove Conjecture 3.6, by Lemma 3.5(ii) it is essential to show it for T-invariant divisors $D = \sum w_i D_i$ with $w_i = 0$ or -1.

Remark 3.7. Bondal announced in [4] that Conjecture 3.6 is true for the case $D = 0$. Although the proof is not available so far, several people have already used this statement (cf. [3, 8, 9, 11]). In this article, we refer this statement as *Bondal's conjecture.* Bondal's conjecture is solved for 2-dimensional toric Deligne–Mumford stacks in [22].

Lemma 3.5 will not be used afterwards, but an idea to solve Bondal's conjecture in §4.2 and 4.3 comes from it.

The following is sometimes powerful when we show that $F_{m*}\mathcal{O}_X$ is a tilting object.

Lemma 3.8. *Take a line bundle $\mathcal{L}$ on X.*

(i) If $\mathcal{L}^{-1}$ is nef, then we have $\mathrm{Ext}^i_X(\mathcal{L}, F_{m*}\mathcal{O}_X) = 0$ for $i > 0$.

(ii) [23] If $\mathcal{L} \otimes \omega_X^{-1}$ is ample, then we have $\mathrm{Ext}^i_X(F_{m*}\mathcal{O}_X, \mathcal{L}) = 0$ for $i > 0$.

Proof. (i) By adjunction, we have $\mathrm{Ext}^i_X(\mathcal{L}, F_{m*}\mathcal{O}_X) = H^i(X, \mathcal{L}^{-m})$. But the last term vanishes for $i > 0$, since X is toric.

(ii) We have $\mathrm{Ext}^i_X(F_{m*}\mathcal{O}_X, \mathcal{L}) = H^i(X, F_m^*(\mathcal{L} \otimes \omega_X^{-1}) \otimes \omega_X)$, which vanishes by the Kodaira vanishing theorem. □

We have the following easy lemma. Because of it, the facts that the set $\mathfrak{D}_X$ forms a full strong exceptional collection and that $F_{m*}\mathcal{O}_X$ is a tilting generator for sufficiently large m are equivalent.

Lemma 3.9. *(i) Let us consider a finite set of line bundles $\{\mathcal{L}_k\}$ satisfying $\mathcal{L}_i \not\cong \mathcal{L}_j$ for $i \neq j$. Assume that the vector bundle $\mathcal{E} = \bigoplus \mathcal{L}_k$ is a* tilting generator *of $D^b(X)$, namely it satisfies the following conditions:*

(1) $\mathrm{Hom}^i_X(\mathcal{E}, \mathcal{E}) = 0$ for $i \neq 0$. Such an object $\mathcal{E}$ in $D^b(X)$ is called a tilting *object.*

(2) $\langle \mathcal{E} \rangle^\perp = 0$ in $D^b(X)$, that is, $\mathcal{E}$ is a generator *of $D^b(X)$.*

Then the set $\{\mathcal{L}_i\}$ forms a full strong exceptional collection.

(ii) Suppose that we have a full strong exceptional collection $\{\mathcal{L}_k\}$ on X. Then their direct sum $\bigoplus_k \mathcal{L}_k$ is a tilting generator of $D^b(X)$.

Proof. The most parts of the statements are direct consequences of the definitions. I explain only how to show the fullness in (i).

Note that the condition (1) implies that the set $\{\mathcal{L}_k\}$ is a strong exceptional collection. Then we have a semi-orthogonal decomposition (cf. [16, page 25]) of $D^b(X)$ into $\langle \{\mathcal{L}_k\} \rangle^\perp$ and $\langle \{\mathcal{L}_k\} \rangle$. Since $\mathcal{E}$ is a generator, $\langle \{\mathcal{L}_k\} \rangle^\perp = 0$ which implies that the strong exceptional collection $\{\mathcal{L}_k\}$ is full. □

4 Examples

In this section, we determine the set $\mathfrak{D}$ for various smooth toric varieties.

4.1 Maximal toric del Pezzo surface

Let us consider the toric surface $X = Y_3$ which is obtained by the blow up of $\mathbb{P}^2$ at the three T-invariant points. Namely, X is the *maximal* toric del Pezzo surface with respect to birational relations. We put

$$\begin{aligned} \boldsymbol{v}_1 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \boldsymbol{v}_3 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \\ \boldsymbol{v}_4 &= \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \boldsymbol{v}_5 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \boldsymbol{v}_6 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \in \mathbb{Z}^2. \end{aligned}$$

Then we know that D_2, D_4, D_6 are exceptional divisors of $X \to \mathbb{P}^2$. Note that $D_1 + D_6 \sim D_3 + D_4$ and $D_2 + D_3 \sim D_5 + D_6$.

For $\boldsymbol{u} = \begin{pmatrix} x \\ y \end{pmatrix} \in P^2_m$, we have

$$\boldsymbol{q}^m(\boldsymbol{u}) = \begin{pmatrix} \lfloor \frac{x}{m} \rfloor \\ \lfloor \frac{y}{m} \rfloor \\ \lfloor \frac{-x+y}{m} \rfloor \\ \lfloor \frac{-x}{m} \rfloor \\ \lfloor \frac{-y}{m} \rfloor \\ \lfloor \frac{x-y}{m} \rfloor \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \lfloor \frac{-x+y}{m} \rfloor \\ \lfloor \frac{-x}{m} \rfloor \\ \lfloor \frac{-y}{m} \rfloor \\ \lfloor \frac{x-y}{m} \rfloor \end{pmatrix}.$$

Then we obtain

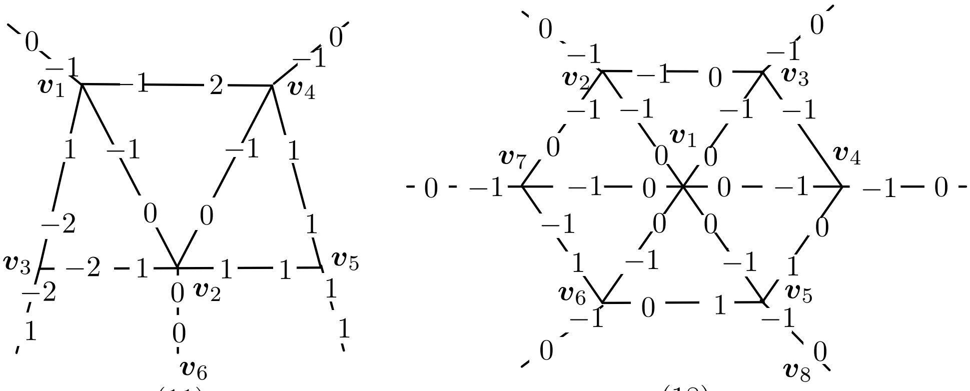
$$\begin{aligned} \mathfrak{D} = \big\{ &\mathcal{O}_X(-D_5 - D_6), \mathcal{O}_X(-D_3 - D_4), \mathcal{O}_X(-D_4 - D_5), \\ &\mathcal{O}_X(-D_3 - D_4 - D_5), \mathcal{O}_X(-D_4 - D_5 - D_6), \mathcal{O}_X \big\}. \end{aligned}$$

These are dual to the line bundles which appear in a full strong exceptional collections on X in [19]. In particular, $\mathfrak{D}$ forms a full strong exceptional collection.

4.2 Fano 3-fold in (11)

Take the Fano 3-fold X in (11). Put

$$\begin{aligned} \boldsymbol{v}_1 &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \boldsymbol{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \\ \boldsymbol{v}_4 &= \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \boldsymbol{v}_5 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \boldsymbol{v}_6 = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} \in \mathbb{Z}^2, \end{aligned}$$



(11) (18)

Figure 5: Fano 3-folds in (11) and (18)

as in Figure 5.

For $\boldsymbol{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in P_m^3$, we have

$$\boldsymbol{q}^m(\boldsymbol{u}) = \begin{pmatrix} \lfloor \frac{x}{m} \rfloor \\ \lfloor \frac{y}{m} \rfloor \\ \lfloor \frac{z}{m} \rfloor \\ \lfloor \frac{x-z}{m} \rfloor \\ \lfloor \frac{-z}{m} \rfloor \\ \lfloor \frac{-x-y+2z}{m} \rfloor \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \lfloor \frac{x-z}{m} \rfloor \\ \lfloor \frac{-z}{m} \rfloor \\ \lfloor \frac{-x-y+2z}{m} \rfloor \end{pmatrix}.$$

Therefore we have

$$\begin{aligned}\mathfrak{D} = \big\{&\mathcal{O}_X, \mathcal{O}_X(-D_6), \mathcal{O}_X(-2D_6), \mathcal{O}_X(-D_5), \mathcal{O}_X(-D_5 - D_6), \mathcal{O}_X(-D_5 - 2D_6),\\ &\mathcal{O}_X(-D_4 - D_5 - D_6), \mathcal{O}_X(-D_4 - D_5), \mathcal{O}_X(-D_4 - D_5 + D_6)\big\}.\end{aligned}$$

Then by the equation (1) we can read from Figure 5 that for all $\mathcal{L} \in \mathfrak{D}$ except $\mathcal{O}_X(-D_4 - D_5 + D_6)$, $\mathcal{L}^{-1}$ is nef, hence Lemma 3.8 implies that $\mathrm{Ext}^i_X(\mathcal{L}, F_{m*}\mathcal{O}_X) = 0$ for $i > 0$. Put

$$D_{nef} := \mathfrak{D} \backslash \{\mathcal{O}_X(-D_4 - D_5 + D_6)\}.$$

We shall prove in §5.2 that $\langle \mathfrak{D}_{nef} \rangle = D^b(X)$. Consequently the set $\mathfrak{D}_{nef}$ becomes a full strong exceptional collection.

4.3 Fano 3-fold in (18)

Take the Fano 3-fold in (18). Put

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \boldsymbol{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \boldsymbol{v}_4 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix},$$
$$\boldsymbol{v}_5 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \boldsymbol{v}_6 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \boldsymbol{v}_7 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \boldsymbol{v}_8 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \in \mathbb{Z}^2,$$

as in Figure 5. For $\boldsymbol{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in P_m^3$, we have

$$\boldsymbol{q}^m(\boldsymbol{u}) = \begin{pmatrix} \lfloor \frac{x}{m} \rfloor \\ \lfloor \frac{y}{m} \rfloor \\ \lfloor \frac{z}{m} \rfloor \\ \lfloor \frac{-y+z}{m} \rfloor \\ \lfloor \frac{-y}{m} \rfloor \\ \lfloor \frac{-z}{m} \rfloor \\ \lfloor \frac{y-z}{m} \rfloor \\ \lfloor \frac{-x+y}{m} \rfloor \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \lfloor \frac{-y+z}{m} \rfloor \\ \lfloor \frac{-y}{m} \rfloor \\ \lfloor \frac{-z}{m} \rfloor \\ \lfloor \frac{y-z}{m} \rfloor \\ \lfloor \frac{-x+y}{m} \rfloor \end{pmatrix}.$$

Therefore we have

$$\begin{aligned} \mathfrak{D} = \big\{ & \mathcal{O}_X(-iD_8), \mathcal{O}_X(-D_6 - D_7 - iD_8), \mathcal{O}_X(-D_4 - D_5 - iD_8), \\ & \mathcal{O}_X(-D_5 - D_6 - D_7 - iD_8), \mathcal{O}_X(-D_5 - D_6 - iD_8), \\ & \mathcal{O}_X(-D_4 - D_5 - D_6 - iD_8) \bigm| i = 0, 1 \big\}. \end{aligned}$$

By the equation (1), we can read from Figure 5 that that $\mathcal{L}^{-1}$ is nef for all $\mathcal{L} \in \mathfrak{D}$. Hence by Lemma 3.8 implies that $\mathrm{Ext}^i_X(F_{m*}\mathcal{O}_X, F_{m*}\mathcal{O}_X) = 0$ for all $m \gg 0, i > 0$.

4.4 Fano 3-fold in (8)

Take the Fano 3-fold X in (8). Put

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \boldsymbol{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$
$$\boldsymbol{v}_4 = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}, \boldsymbol{v}_5 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \boldsymbol{v}_6 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{Z}^2.$$

For $\boldsymbol{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in P_m^3$, we have

$$\boldsymbol{q}^m(\boldsymbol{u}) = \begin{pmatrix} \lfloor \frac{x}{m} \rfloor \\ \lfloor \frac{y}{m} \rfloor \\ \lfloor \frac{z}{m} \rfloor \\ \lfloor \frac{-x-z}{m} \rfloor \\ \lfloor \frac{x-y}{m} \rfloor \\ \lfloor \frac{-x}{m} \rfloor \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \lfloor \frac{-x-z}{m} \rfloor \\ \lfloor \frac{x-y}{m} \rfloor \\ \lfloor \frac{-x}{m} \rfloor \end{pmatrix}.$$

Therefore we have

$$\begin{aligned}\mathfrak{D} = \big\{ &\mathcal{O}_X, \mathcal{O}_X(-D_5), \mathcal{O}_X(-D_4), \mathcal{O}_X(-D_4 - D_5), \mathcal{O}_X(-D_4 - D_6), \\ &\mathcal{O}_X(-D_4 - D_5 - D_6), \mathcal{O}_X(-2D_4 - D_6), \mathcal{O}_X(-2D_4 - D_5 - D_6) \big\}.\end{aligned}$$

For all of line bundles $\mathcal{L} \in \mathfrak{D}$, we can see that $\mathcal{L}^{-1}$ is nef by a similar method to one above, hence Lemma 3.8 implies that $\mathrm{Ext}^i_X(F_{m*}\mathcal{O}_X, F_{m*}\mathcal{O}_X) = 0$ for $i > 0$. This result contradicts the result in [23, page 32].

5 Exceptional collections on maximal toric Fano 3-folds

In this section, we prove Bondal's conjecture for maximal smooth toric Fano 3-folds. Combining this with the results in §4, we see that $\mathfrak{D}_X$ (respectively, $\mathfrak{D}_{nef}$) is a full strong exceptional collection in the cases Fano 3-folds in (17) and (18) (respectively, (11)).

Lemma 5.1. *Let $f\colon X \to Y$ be a proper morphism between smooth varieties. Suppose that an object $\mathcal{E}$ is a generator of $D^b(X)$ and $\mathcal{O}_Y$ is a direct summand of the object $\mathbb{R}f_*\mathcal{O}_X$. Then $\mathbb{R}f_*\mathcal{E}$ is also a generator of $D^b(Y)$.*

Proof. Put $\mathbb{R}\mathrm{Hom}_Y(\mathbb{R}f_*\mathcal{E}, \mathcal{F}) = 0$ for some $\mathcal{F} \in D^b(Y)$. Then by the adjointness $\mathbb{R}f_* \dashv f^!$, we obtain $\omega_X \otimes \mathbb{L}f^*(\mathcal{F} \otimes \omega_Y^{-1}) = f^!\mathcal{F} = 0$, which implies that $\mathbb{L}f^*\mathcal{F} = 0$. Since $\mathcal{F}$ is a direct summand of $\mathbb{R}f_*\mathbb{L}f^*\mathcal{F} = \mathcal{F} \overset{\mathbb{L}}{\otimes} \mathbb{R}f_*\mathcal{O}_X$, we obtain the assertion. □

In Lemma 5.1, main examples in mind are the following: Let X and Y be smooth projective toric varieties.

(i) For the toric blow up $f\colon X \to Y$, by $\mathbb{R}f_*\mathcal{O}_X = \mathcal{O}_Y$ and the commutativity $F_m^Y \circ f = f \circ F_m^X$, we have $\mathbb{R}f_*F^X_{m*}\mathcal{O}_X = F^Y_{m*}\mathcal{O}_Y$. Hence if $F_{m*}O_X$ is a generator, then so is $F_{m*}\mathcal{O}_Y$.

(ii) For the Frobenius morphism F_m on X, $F_{m*}\mathcal{E}$ is a generator of $D^b(X)$ for a generator $\mathcal{E}$ of $D^b(X)$. Note that $\mathcal{O}_X$ is indeed a direct summand of $F_{m*}\mathcal{O}_X$.

5.1 Exceptional collection on the Fano 3-fold in (17)

Let us consider the Fano 3-fold X in (17) which is the product of the maximal toric del Pezzo surface Y_3 in §4.1 and a projective line $\mathbb{P}^1$. The following must be well-known.

Lemma 5.2. *Let Y and Z be smooth projective varieties. Suppose that $\mathcal{E}$ and $\mathcal{F}$ are tilting generators of $D^b(Y)$ and $D^b(Z)$ respectively. Then $\mathcal{E} \overset{\mathbb{L}}{\boxtimes} \mathcal{F}$ is also a tilting generator of $D^b(Y \times Z)$.*

Proof. We can check

$$\mathbb{R}\Gamma(Y \times Z, \mathcal{E} \overset{\mathbb{L}}{\boxtimes} \mathcal{F}) = \mathbb{R}\Gamma(Y, \mathcal{E}) \overset{\mathbb{L}}{\otimes} \mathbb{R}\Gamma(Z, \mathcal{F}).$$

Hence we have

$$\mathrm{Hom}^i_{Y\times Z}(\mathcal{E} \overset{\mathbb{L}}{\boxtimes} \mathcal{F}, \mathcal{E} \overset{\mathbb{L}}{\boxtimes} \mathcal{F}) \cong \bigoplus_{j+k=i} \mathrm{Hom}^j_Y(\mathcal{E}, \mathcal{E}) \otimes \mathrm{Hom}^k_Z(\mathcal{F}, \mathcal{F}),$$

which implies that $\mathcal{E} \overset{\mathbb{L}}{\boxtimes} \mathcal{F}$ is tilting. The fact $\mathcal{E} \overset{\mathbb{L}}{\boxtimes} \mathcal{F}$ is a generator directly follows from [6, Lemma 3.4.1]. $\square$

For the toric case, we know that $F^{Y\times Z}_{m*}\mathcal{O}_{Y\times Z} \cong F^Y_{m*}\mathcal{O}_Y \boxtimes F^Z_{m*}\mathcal{O}_Z$. Moreover we see in §4.1 that $F_{m*}\mathcal{O}_{Y_3}$ is a tilting generator on the maximal toric del Pezzo surface Y_3 and $m \gg 0$. It follows that $\mathfrak{D}_X$ is a full strong exceptional collection. Here we leave to readers the proof of the fact that $\mathfrak{D}_{\mathbb{P}^1}$ is a full strong exceptional collection on $\mathbb{P}^1$.

5.2 Exceptional collection on Fano 3-fold in (11)

Take the Fano 3-fold X in (11). We use the same notation as in §4.2. Let us recall that

$$\begin{aligned}\mathfrak{D}_{nef} = \big\{&\mathcal{O}_X, \mathcal{O}_X(-D_6), \mathcal{O}_X(-2D_6), \mathcal{O}_X(-D_5), \mathcal{O}_X(-D_5 - D_6),\\ &\mathcal{O}_X(-D_5 - 2D_6), \mathcal{O}_X(-D_4 - D_5 - D_6), \mathcal{O}_X(-D_4 - D_5)\big\}.\end{aligned}$$

The next is the aim of §5.2.

Proposition 5.3. *Let X be the toric Fano 3-fold in (11). Then the set $\mathfrak{D}_{nef}$ forms a full strong exceptional collection.*

First we determine the set $\mathfrak{D}(\omega_X^{-3})_m$ for sufficiently large m. For $\boldsymbol{u} =$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in P^3_m$ and $\boldsymbol{w} = \begin{pmatrix} -1 \\ \vdots \\ -1 \end{pmatrix} \in \mathbb{Z}^6$, we have

$$\boldsymbol{q}^m(\boldsymbol{u}, -3\boldsymbol{w}) = \begin{pmatrix} \lfloor \frac{(x-3)+3}{m} \rfloor \\ \lfloor \frac{(y-3)+3}{m} \rfloor \\ \lfloor \frac{(z-3)+3}{m} \rfloor \\ \lfloor \frac{(x-3)-(z-3)+3}{m} \rfloor \\ \lfloor \frac{-(z-3)+3}{m} \rfloor \\ \lfloor \frac{-(x-3)-(y-3)+2(z-3)+3}{m} \rfloor \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \lfloor \frac{x-z+3}{m} \rfloor \\ \lfloor \frac{-z+6}{m} \rfloor \\ \lfloor \frac{-x-y+2z+3}{m} \rfloor \end{pmatrix}.$$

Thus we have

$$q_4^m(\boldsymbol{u}, -3\boldsymbol{w}) = \begin{cases} 1 & \text{if } x+3 \geq z+m \\ 0 & \text{if } z+m > x+3 \geq z \\ -1 & \text{if } z > x+3 \end{cases}$$

$$q_5^m(\boldsymbol{u}, -3\boldsymbol{w}) = \begin{cases} 0 & \text{if } 6 \geq z \\ -1 & \text{if } z > 6 \end{cases}$$

$$q_6^m(\boldsymbol{u}, -3\boldsymbol{w}) = \begin{cases} 2 & \text{if } 2z+3 \geq x+y+2m \\ 1 & \text{if } x+y+2m > 2z+3 \geq x+y+m \\ 0 & \text{if } x+y+m > 2z+3 \geq x+y \\ -1 & \text{if } x+y > 2z+3 \geq x+y-m \\ -2 & \text{if } x+y-m > 2z+3. \end{cases}$$

By tedious computation, we can see $\mathfrak{D}(\omega_X^{-3})_m = \mathfrak{D} \cup \mathfrak{D}'$, where

$$\begin{aligned} \mathfrak{D}' = \big\{ &\mathcal{O}_X(-D_4 - D_5 + 2D_6), \mathcal{O}_X(-D_5 + D_6), \\ &\mathcal{O}_X(-D_4 - iD_6), \mathcal{O}_X(D_4 - jD_6) \;\big|\; i = 0, 1 \text{ and } j = 1, 2 \big\}. \end{aligned}$$

Note that there are linear equivalences;

$$D_1 + D_4 \sim D_6, \quad D_2 \sim D_6, \quad D_3 + 2D_6 \sim D_4 + D_5. \tag{6}$$

Claim 5.4. $\langle \mathfrak{D}_{nef} \rangle = \big\langle \mathfrak{D}(\omega_X^{-3})_m \big\rangle$.

Proof. We shall check that $\mathcal{L} \in \langle \mathfrak{D}_{nef} \rangle$ for $\mathcal{L} = \mathcal{O}_X(-D_4 - D_5 + D_6)$ and all $\mathcal{L} \in \mathfrak{D}'$ below. Note that this implies that $\langle \mathfrak{D}_{nef} \rangle = \big\langle \mathfrak{D}(\omega_X^{-3})_m \big\rangle$, since

$$\mathfrak{D}(\omega_X^{-3})_m = \mathfrak{D} \cup \mathfrak{D}' = \mathfrak{D}_{nef} \cup \{\mathcal{O}_X(-D_4 - D_5 + D_6)\} \cup \mathfrak{D}'.$$

Since $D_1 \cap D_2 \cap D_6 = \emptyset$, we have an exact sequence

$$\begin{aligned} 0 \to &\mathcal{O}_X(-D_1 - D_2 - D_6) \\ \to &\mathcal{O}_X(-D_1 - D_2) \oplus \mathcal{O}_X(-D_2 - D_6) \oplus \mathcal{O}_X(-D_1 - D_6) \\ \to &\mathcal{O}_X(-D_1) \oplus \mathcal{O}_X(-D_2) \oplus \mathcal{O}_X(-D_6) \to \mathcal{O}_X \to 0. \end{aligned}$$

Combining this with (6), we have an exact sequence

$$\begin{aligned}0 \to &\mathcal{O}_X(D_4 - 3D_6) \to \mathcal{O}_X(D_4 - 2D_6)^{\oplus 2} \oplus \mathcal{O}_X(-2D_6) \\ \to &\mathcal{O}_X(D_4 - D_6) \oplus \mathcal{O}_X(-D_6)^{\oplus 2} \to \mathcal{O}_X \to 0. \end{aligned} \tag{7}$$

Similarly, since $D_2 \cap D_4 \cap D_6 = \emptyset$, we have an exact sequence

$$\begin{aligned}0 \to &\mathcal{O}_X(-D_4 - 2D_6) \to \mathcal{O}_X(-D_4 - D_6)^{\oplus 2} \oplus \mathcal{O}_X(-2D_6) \\ \to &\mathcal{O}_X(-D_4) \oplus \mathcal{O}_X(-D_6)^{\oplus 2} \to \mathcal{O}_X \to 0. \end{aligned} \tag{8}$$

Since $D_3 \cap D_4 = \emptyset$, we have an exact sequence

$$0 \to \mathcal{O}_X(-D_3 - D_4) \to \mathcal{O}_X(-D_3) \oplus \mathcal{O}_X(-D_4) \to \mathcal{O}_X \to 0.$$

Using (6), we have an exact sequence

$$\begin{aligned}0 \to &\mathcal{O}_X(-2D_4 - D_5 + 2D_6) \\ \to &\mathcal{O}_X(-D_4 - D_5 + 2D_6) \oplus \mathcal{O}_X(-D_4) \to \mathcal{O}_X \to 0. \end{aligned} \tag{9}$$

Similarly, since $D_1 \cap D_5 = \emptyset$, we have an exact sequence

$$\begin{aligned}0 \to &\mathcal{O}_X(D_4 - D_5 - D_6) \\ \to &\mathcal{O}_X(D_4 - D_6) \oplus \mathcal{O}_X(-D_5) \to \mathcal{O}_X \to 0. \end{aligned} \tag{10}$$

(i) Tensoring $\mathcal{O}_X(-D_4 - D_5 + D_6)$ with (7), we obtain an exact sequence

$$\begin{aligned}0 \to &\mathcal{O}_X(-D_5 - 2D_6) \to \mathcal{O}_X(-D_5 - D_6)^{\oplus 2} \oplus \mathcal{O}_X(-D_4 - D_5 - D_6) \\ \to &\mathcal{O}_X(-D_5) \oplus \mathcal{O}_X(-D_4 - D_5)^{\oplus 2} \to \mathcal{O}_X(-D_4 - D_5 + D_6) \to 0. \end{aligned}$$

We have already known that all line bundles in the sequence except $\mathcal{O}_X(-D_4 - D_5 + D_6)$ belong to $\langle \mathfrak{D}_{nef} \rangle$. Thus so does $\mathcal{O}_X(-D_4 - D_5 + D_6)$.[3]

(ii) Tensoring $\mathcal{O}_X(-D_5 + D_6)$ with (8), we obtain an exact sequence

$$\begin{aligned}0 \to &\mathcal{O}_X(-D_4 - D_5 - D_6) \to \mathcal{O}_X(-D_4 - D_5)^{\oplus 2} \oplus \mathcal{O}_X(-D_5 - D_6) \\ \to &\mathcal{O}_X(-D_4 - D_5 + D_6) \oplus \mathcal{O}_X(-D_5)^{\oplus 2} \to \mathcal{O}_X(-D_5 + D_6) \to 0. \end{aligned}$$

We have already known from (i) that all line bundles in the sequence except $\mathcal{O}_X(-D_5 + D_6)$ belong to $\langle \mathfrak{D}_{nef} \rangle$. Thus so does $\mathcal{O}_X(-D_5 + D_6)$.

(iii) Tensoring $\mathcal{O}_X(-D_4 - D_5 + 2D_6)$ with (7), we obtain an exact sequence

$$\begin{aligned}0 \to &\mathcal{O}_X(-D_5 - D_6) \to \mathcal{O}_X(-D_5)^{\oplus 2} \oplus \mathcal{O}_X(-D_4 - D_5) \\ \to &\mathcal{O}_X(-D_4 - D_5 + D_6)^{\oplus 2} \oplus \mathcal{O}_X(-D_5 + D_6) \to \mathcal{O}_X(-D_4 - D_5 + 2D_6) \to 0. \end{aligned}$$

[3] This proves the fact $\langle \mathfrak{D}_{nef} \rangle = \langle \mathfrak{D} \rangle$, which has been already observed in [3, Proposition 3.2].

We have already known from (i) and (ii) that all line bundles in the sequence except $\mathcal{O}_X(-D_4 - D_5 + 2D_6)$ belong to $\langle \mathfrak{D}_{nef} \rangle$. Thus so does $\mathcal{O}_X(-D_4 - D_5 + 2D_6)$.

(iv) Take $j = 1, 2$. Tensoring $\mathcal{O}_X(D_4 - jD_6)$ with (9), we obtain an exact sequence

$$\begin{aligned} 0 \to &\mathcal{O}_X(-D_4 - D_5 + (2-j)D_6) \\ &\to \mathcal{O}_X(-D_5 + (2-j)D_6) \oplus \mathcal{O}_X(-jD_6) \to \mathcal{O}_X(D_4 - jD_6) \to 0. \end{aligned}$$

We have already known from (i) and (ii) that all line bundles in the sequence except $\mathcal{O}_X(D_4 - jD_6)$ belong to $\langle \mathfrak{D}_{nef} \rangle$. Thus so does $\mathcal{O}_X(D_4 - jD_6)$.

(v) Take $i = 0, 1$. Tensoring $\mathcal{O}_X(-D_4 - iD_6)$ with (10), we obtain an exact sequence

$$\begin{aligned} 0 \to &\mathcal{O}_X(-D_5 - (i+1)D_6) \\ &\to \mathcal{O}_X(-(i+1)D_6) \oplus \mathcal{O}_X(-D_4 - D_5 - iD_6) \to \mathcal{O}_X(-D_4 - iD_6) \to 0. \end{aligned}$$

We have already known that all line bundles in the sequence except $\mathcal{O}_X(-D_4 - iD_6)$ belong to $\langle \mathfrak{D}_{nef} \rangle$. Thus so does $\mathcal{O}_X(-D_4 - iD_6)$.

Therefore we know that $\langle \mathfrak{D}_{nef} \rangle = \left\langle \mathfrak{D}(\omega_X^{-3})_m \right\rangle$. □

Now we can prove Proposition 5.3.

Proof. We directly see by computation that

$$\mathfrak{D}_{nef} \subset \mathfrak{D}(\omega_X^{-1})_m \subset \mathfrak{D}(\omega_X^{-2})_m \subset \mathfrak{D}(\omega_X^{-3})_m,$$

which is more or less expected by Lemma 3.5. It is also known that $\bigoplus_{i=0}^{3} \omega_X^{-i}$ is a generator ([25, Lemma 3.2.2]), since ω_X^{-1} is very ample. Lemma 5.1 implies that $\left\langle \mathfrak{D}(\omega_X^{-3})_m \right\rangle^{\perp} = 0$. Thus we can see from Claim 5.4 that

$$\langle \mathfrak{D}_{nef} \rangle^{\perp} = 0.$$

Combining the result in §4.2 with Lemma 3.9(i), we complete the proof. □

5.3 Exceptional collection on Fano 3-fold in (18)

Take the Fano 3-fold in (18), and use the same notation as in §4.3. Recall that

$$\begin{aligned} \mathfrak{D} = \big\{ &\mathcal{O}_X(-iD_8), \mathcal{O}_X(-D_6 - D_7 - iD_8), \mathcal{O}_X(-D_4 - D_5 - iD_8), \\ &\mathcal{O}_X(-D_5 - D_6 - D_7 - iD_8), \mathcal{O}_X(-D_5 - D_6 - iD_8), \\ &\mathcal{O}_X(-D_4 - D_5 - D_6 - iD_8) \bigm| i = 0, 1 \big\}. \end{aligned}$$

We can prove the following.

Proposition 5.5. *Let X be the toric Fano 3-fold in (18). Then the set $\mathfrak{D}$ forms a full strong exceptional collection.*

First we want to find all elements of $\mathfrak{D}(\omega_X^{-3})_m$ for sufficiently large m. For $\boldsymbol{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in P_m^3$ and $\boldsymbol{w} = \begin{pmatrix} -1 \\ \vdots \\ -1 \end{pmatrix} \in \mathbb{Z}^8$, we have

$$\boldsymbol{q}^m(\boldsymbol{u}, -3\boldsymbol{w}) = \begin{pmatrix} \lfloor \frac{(x-3)+3}{m} \rfloor \\ \lfloor \frac{(y-3)+3}{m} \rfloor \\ \lfloor \frac{(z-3)+3}{m} \rfloor \\ \lfloor \frac{-(y-3)+(z-3)+3}{m} \rfloor \\ \lfloor \frac{-(y-3)+3}{m} \rfloor \\ \lfloor \frac{-(z-3)+3}{m} \rfloor \\ \lfloor \frac{(y-3)-(z-3)+3}{m} \rfloor \\ \lfloor \frac{-(x-3)+(y-3)+3}{m} \rfloor \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \lfloor \frac{-y+z+3}{m} \rfloor \\ \lfloor \frac{-y+6}{m} \rfloor \\ \lfloor \frac{-z+6}{m} \rfloor \\ \lfloor \frac{y-z+3}{m} \rfloor \\ \lfloor \frac{-x+y+3}{m} \rfloor \end{pmatrix}.$$

Thus we have

$$q_4^m(\boldsymbol{u}, -3\boldsymbol{w}) = \begin{cases} 1 & \text{if } z \geq y+m-3 \\ 0 & \text{if } y+m-3 > z \geq y-3 \\ -1 & \text{if } y > z+3 \end{cases}$$

$$q_5^m(\boldsymbol{u}, -3\boldsymbol{w}) = \begin{cases} 0 & \text{if } 6 \geq y \\ -1 & \text{if } y > 6 \end{cases}$$

$$q_6^m(\boldsymbol{u}, -3\boldsymbol{w}) = \begin{cases} 0 & \text{if } 6 \geq z \\ -1 & \text{if } z > 6 \end{cases}$$

$$q_7^m(\boldsymbol{u}, -3\boldsymbol{w}) = \begin{cases} 1 & \text{if } y \geq z+m-3 \\ 0 & \text{if } z+m-3 > y \geq z-3 \\ -1 & \text{if } z > y+3 \end{cases}$$

$$q_8^m(\boldsymbol{u}, -3\boldsymbol{w}) = \begin{cases} 1 & \text{if } y \geq x+m-3 \\ 0 & \text{if } x+m-3 > y \geq x-3 \\ -1 & \text{if } x > y+3. \end{cases}$$

Hence by tedious computation, we can see $\mathfrak{D}(\omega_X^{-3})_m = \mathfrak{D} \cup \mathfrak{D}'$, where

$$\begin{aligned} \mathfrak{D}' = \big\{ &\mathcal{O}_X(-D_4 - iD_8), \mathcal{O}_X(-D_5 - iD_8), \mathcal{O}_X(-D_6 - iD_8), \mathcal{O}_X(-D_7 - iD_8), \\ &\mathcal{O}_X(-D_4 - D_5 + D_7 + iD_8), \mathcal{O}_X(D_4 - D_6 - D_7 - iD_8), \\ &\mathcal{O}_X(-D_4 - D_5 + D_8), \mathcal{O}_X(-D_5 - D_6 + D_8), \\ &\mathcal{O}_X(-D_4 - D_5 - D_6 + D_8) \;\big|\; i = 0, 1 \big\}. \end{aligned}$$

Since $\mathfrak{D}$ contains several line bundles of the form $\mathcal{L}$ and $\mathcal{L} \otimes \mathcal{O}_X(D_8)$, Claim 5.6 gives that $\mathcal{L} \otimes \mathcal{O}_X(iD_8) \in \langle \mathfrak{D} \rangle$ for all $i \in \mathbb{Z}$.

Claim 5.6. *If $\mathcal{L}$ and $\mathcal{L} \otimes \mathcal{O}_X(D_8) \in \langle \mathfrak{D} \rangle$, then we have $\mathcal{L} \otimes \mathcal{O}_X(iD_8) \in \langle \mathfrak{D} \rangle$ for all $i \in \mathbb{Z}$.*

Proof. Note that there are linearly equivalences;

$$D_1 \sim D_8, \quad D_2 + D_4 + D_5 \sim D_7 + D_8, \quad D_3 + D_4 \sim D_6 + D_7. \tag{11}$$

We have

$$0 \to \mathcal{O}_X(-D_1 - D_8) \to \mathcal{O}_X(-D_1) \oplus \mathcal{O}_X(-D_8) \to \mathcal{O}_X \to 0.$$

Combining this with (11), we have

$$0 \to \mathcal{O}_X(-2D_8) \to \mathcal{O}_X(-D_8) \oplus \mathcal{O}_X(-D_8) \to \mathcal{O}_X \to 0.$$

By tensoring $\mathcal{L} \in \operatorname{Pic} X$, we obtain the claim. $\square$

Claim 5.7. $\langle \mathfrak{D} \rangle = \left\langle \mathfrak{D}(\omega_X^{-3})_m \right\rangle$.

Proof. We shall check that $\mathcal{L} \in \langle \mathfrak{D} \rangle$ for all $\mathcal{L} \in \mathfrak{D}'$ below. First note that Claim 5.6 implies the last three line bundles in $\mathfrak{D}'$ belong to $\langle \mathfrak{D} \rangle$. We take an arbitrary integer $i \in \mathbb{Z}$ below.

(i) We have exact sequences:

$$\begin{aligned} 0 \to \mathcal{O}_X(-D_5 - D_6 - D_7 + iD_8) \to &\mathcal{O}_X(-D_6 - D_7 + iD_8) \\ &\to \mathcal{O}_{D_5}(-D_6 + iD_8) \to 0, \end{aligned}$$

$$0 \to \mathcal{O}_X(-D_5 - D_6 + iD_8) \to \mathcal{O}_X(-D_6 + iD_8) \to \mathcal{O}_{D_5}(-D_6 + iD_8) \to 0.$$

Hence $\mathcal{O}_X(-D_6 + iD_8) \in \langle \mathfrak{D} \rangle$, since

$$\mathcal{O}_X(-D_5 - D_6 - D_7 + iD_8), \mathcal{O}_X(-D_5 - D_6 + iD_8), \mathcal{O}_X(-D_6 - D_7 + iD_8) \in \langle \mathfrak{D} \rangle.$$

Similarly we obtain $\mathcal{O}_X(-D_5 + iD_8) \in \langle \mathfrak{D} \rangle$.

(ii) We have exact sequences:

$$\begin{aligned} 0 \to \mathcal{O}_X(-D_3 - D_4 - D_5 + iD_8) \to &\mathcal{O}_X(-D_3 - D_4 + iD_8) \\ &\to \mathcal{O}_{D_5}(-D_4 + iD_8) \to 0, \end{aligned}$$

$$0 \to \mathcal{O}_X(-D_4 - D_5 + iD_8) \to \mathcal{O}_X(-D_4 + iD_8) \to \mathcal{O}_{D_5}(-D_4 + iD_8) \to 0.$$

Since the line bundles

$$\mathcal{O}_X(-D_3 - D_4 - D_5 + iD_8) \cong \mathcal{O}_X(-D_5 - D_6 - D_7 + iD_8),$$

$$\mathcal{O}_X(-D_3 - D_4 + iD_8) \cong \mathcal{O}_X(-D_6 - D_7 + iD_8)$$

and

$$\mathcal{O}_X(-D_4 - D_5 + iD_8)$$

belong to $\langle \mathfrak{D} \rangle$ by (11), we have $\mathcal{O}_X(-D_4 + iD_8) \in \langle \mathfrak{D} \rangle$.

(iii) We have exact sequences:

$$\begin{aligned} 0 \to \mathcal{O}_X(-D_2 - D_6 - D_7 + iD_8) \to &\mathcal{O}_X(-D_2 - D_7 + iD_8) \\ &\to \mathcal{O}_{D_6}(-D_7 + iD_8) \to 0, \end{aligned}$$

$$0 \to \mathcal{O}_X(-D_6 - D_7 + iD_8) \to \mathcal{O}_X(-D_7 + iD_8) \to \mathcal{O}_{D_6}(-D_7 + iD_8) \to 0.$$

Since we can see from (11) that

$$\begin{aligned} \mathcal{O}_X(-D_2 - D_6 - D_7 + iD_8) \cong \mathcal{O}_X(-D_4 - D_5 - D_6 + (i+1)D_8) \in \langle \mathfrak{D} \rangle , \\ \mathcal{O}_X(-D_2 - D_7 + iD_8) \cong \mathcal{O}_X(-D_5 - D_6 + (i+1)D_8) \in \langle \mathfrak{D} \rangle , \\ \mathcal{O}_X(-D_6 - D_7 + iD_8) \in \langle \mathfrak{D} \rangle , \end{aligned}$$

we know that $\mathcal{O}_X(-D_7 + iD_8) \in \langle \mathfrak{D} \rangle$.

(iv) We have exact sequences:

$$\begin{aligned} 0 \to \mathcal{O}_X(-D_2 - D_3 - D_7 + iD_8) \to &\mathcal{O}_X(-D_2 - D_3 + iD_8) \\ &\to \mathcal{O}_{D_7}(-D_2 + iD_8) \to 0, \end{aligned}$$

$$0 \to \mathcal{O}_X(-D_2 - D_7 + iD_8) \to \mathcal{O}_X(-D_2 + iD_8) \to \mathcal{O}_{D_7}(-D_2 + iD_8) \to 0.$$

Since we have

$$\begin{aligned} \mathcal{O}_X(-D_2 - D_3 - D_7 + iD_8) \cong \mathcal{O}_X(-D_5 - D_6 - D_7 + (i+1)D_8) \in \langle \mathfrak{D} \rangle , \\ \mathcal{O}_X(-D_2 - D_3 + iD_8) \cong \mathcal{O}_X(-D_5 - D_6 + (i+1)D_8) \in \langle \mathfrak{D} \rangle , \\ \mathcal{O}_X(-D_2 - D_7 + iD_8) \cong \mathcal{O}_X(-D_4 - D_5 + (i+1)D_8) \in \langle \mathfrak{D} \rangle \end{aligned}$$

by (11), we obtain

$$\mathcal{O}_X(-D_4 - D_5 + D_7 + (i+1)D_8) \cong \mathcal{O}_X(-D_2 + iD_8) \in \langle \mathfrak{D} \rangle .$$

(v) We have exact sequences:

$$\begin{aligned} 0 \to \mathcal{O}_X(-D_2 - D_3 - D_4 + iD_8) \to &\mathcal{O}_X(-D_3 - D_4 + iD_8) \\ &\to \mathcal{O}_{D_2}(-D_3 + iD_8) \to 0, \end{aligned}$$

$$0 \to \mathcal{O}_X(-D_2 - D_3 + iD_8) \to \mathcal{O}_X(-D_3 + iD_8) \to \mathcal{O}_{D_2}(-D_3 + iD_8) \to 0$$

Since we have

$$\begin{aligned} \mathcal{O}_X(-D_2 - D_3 - D_4 + iD_8) \cong \mathcal{O}_X(-D_4 - D_5 - D_6 + (i+1)D_8) \in \langle \mathfrak{D} \rangle , \\ \mathcal{O}_X(-D_3 - D_4 + iD_8) \cong \mathcal{O}_X(-D_6 - D_7 + iD_8) \in \langle \mathfrak{D} \rangle , \\ \mathcal{O}_X(D_2 - D_3 + iD_8) \cong \mathcal{O}_X(-D_5 - D_6 + iD_8) \in \langle \mathfrak{D} \rangle \end{aligned}$$

by (11), we obtain $\mathcal{O}_X(D_4 - D_6 - D_7 + iD_8) \cong \mathcal{O}_X(-D_3 + iD_8) \in \langle \mathfrak{D} \rangle$.

Hence we know that $\langle \mathfrak{D} \rangle = \big\langle \mathfrak{D}(\omega_X^{-3})_m \big\rangle$. $\square$

Then by a similar argument to one given in §5.2, we obtain Proposition 5.5.

6 Birational contractions and tilting objects

Lemma 6.1. *Let $(f,\varphi)\colon (X,\Delta_X)\to(Y,\Delta_Y)$ be a T-equivariant extremal birational contraction between smooth projective toric varieties. Choose a maximal cone σ in Δ_X such that $\varphi(\sigma)$ is a cone in Δ_Y. For any $\boldsymbol{u}\in P^n_m$, we denote a divisor $D^X_{\boldsymbol{u},\mathbf{0},\sigma}$ on X (resp. $D^Y_{\boldsymbol{u},\mathbf{0},\varphi(\sigma)}$ on Y) by $D^X_{\boldsymbol{u}}$ (resp. $D^Y_{\boldsymbol{u}}$). Then we have $f_*\mathcal{O}_X(D^X_{\boldsymbol{u}})=\mathcal{O}_Y(D^Y_{\boldsymbol{u}})$. In particular,*

$$\mathfrak{D}_Y=\left\{f_*\mathcal{L}_X \;\middle|\; \mathcal{L}_X\in\mathfrak{D}_X\right\}.$$

and

$$\mathcal{O}_X(D^X_{\boldsymbol{u}})=f^*\mathcal{O}_Y(D^Y_{\boldsymbol{u}})\otimes\mathcal{O}_X(aE),$$

where $a\geq 0$ and E is the exceptional divisor of f.

Proof. From the commutativity $F_m\circ f=f\circ F_m$, we obtain

$$\bigoplus_{\boldsymbol{u}\in P^n_m}\mathcal{O}_Y(D^Y_{\boldsymbol{u}})=F_{m*}\mathcal{O}_Y=F_{m*}f_*\mathcal{O}_X=f_*F_{m*}\mathcal{O}_X=\bigoplus_{\boldsymbol{u}\in P^n_m}f_*\mathcal{O}_X(D^X_{\boldsymbol{u}}).$$

Fix some $\boldsymbol{u}_1\in P^n_m$. By the above equalities, the canonical inclusion and projection, let us define the maps

$$\alpha_{\boldsymbol{u}'}\colon\mathcal{O}_Y(D^Y_{\boldsymbol{u}_1})\hookrightarrow\bigoplus_{\boldsymbol{u}\in P^n_m}\mathcal{O}_Y(D^Y_{\boldsymbol{u}})=\bigoplus_{\boldsymbol{u}\in P^n_m}f_*\mathcal{O}_X(D^X_{\boldsymbol{u}})\twoheadrightarrow f_*\mathcal{O}_X(D^X_{\boldsymbol{u}'})$$

and

$$\beta_{\boldsymbol{u}'}\colon f_*\mathcal{O}_X(D^X_{\boldsymbol{u}'})\hookrightarrow\bigoplus_{\boldsymbol{u}\in P^n_m}f_*\mathcal{O}_X(D^X_{\boldsymbol{u}})=\bigoplus_{\boldsymbol{u}\in P^n_m}\mathcal{O}_Y(D^Y_{\boldsymbol{u}})\twoheadrightarrow\mathcal{O}_Y(D^Y_{\boldsymbol{u}_1})$$

for each $\boldsymbol{u}'\in P^n_m$. Then we have

$$\sum_{\boldsymbol{u}'\in P^n_m}\beta_{\boldsymbol{u}'}\circ\alpha_{\boldsymbol{u}'}=\mathrm{id},$$

and hence $\beta_{\boldsymbol{u}_2}\circ\alpha_{\boldsymbol{u}_2}\neq 0$ for some $\boldsymbol{u}_2$. Since $\mathrm{End}_Y(\mathcal{O}_Y(D^Y_{\boldsymbol{u}_1}))=\mathbb{C}$, we obtain $f_*\mathcal{O}_X(D^X_{\boldsymbol{u}_2})=\mathcal{O}_Y(D^Y_{\boldsymbol{u}_1})$. Apply a similar argument for

$$\bigoplus_{\boldsymbol{u}\in P^n_m\setminus\{\boldsymbol{u}_2\}}f_*\mathcal{O}_X(D^X_{\boldsymbol{u}})=\bigoplus_{\boldsymbol{u}\in P^n_m\setminus\{\boldsymbol{u}_1\}}\mathcal{O}_Y(D^Y_{\boldsymbol{u}}).$$

Then we can conclude that for all $\boldsymbol{u}\in P^n_m$ we have $\boldsymbol{u}'\in P^n_m$ such that $f_*\mathcal{O}_X(D^X_{\boldsymbol{u}})=\mathcal{O}_Y(D^Y_{\boldsymbol{u}'})$.

On the other hand, we have an inclusion $f_*\mathcal{O}_X(D^X_{\boldsymbol{u}})\hookrightarrow\mathcal{O}_Y(D^Y_{\boldsymbol{u}})$, which is isomorphic in codimension one. Thus it is isomorphic. $\square$

Lemma 6.2. *In the situation of Lemma* 6.1*, assume that f is a equivariant blowing-up along a T-invariant smooth center C, and define $d := \dim E - \dim C$, $n := \dim X$ and $\mathcal{L}_Y := f_*\mathcal{L}_X$ for some $\mathcal{L}_X \in \mathfrak{D}_X$. Consider the Leray spectral sequence*

$$\begin{aligned} E_2^{p,q} =& H^p(Y, \mathcal{L}_Y^{\otimes m} \otimes \omega_Y \otimes \mathbb{R}^q f_* \mathcal{O}_X((ma+d)E)) \\ \Longrightarrow E^{p+q} =& H^{p+q}(X, f^*(\mathcal{L}_Y^{\otimes m} \otimes \omega_X) \otimes \mathcal{O}_X((ma+d)E)) \end{aligned}$$

and assume furthermore that the vanishing

$$\mathrm{Hom}^i_X(\mathcal{L}_X, F_{m*}\mathcal{O}_X) = 0 \tag{12}$$

holds for all $i > 0$.

(i) The vanishing

$$\mathrm{Hom}^i_Y(\mathcal{L}_Y, F_{m*}\mathcal{O}_Y) = 0$$

holds for all $i > 0$ if and only if $E_2^{p,d} = 0$ for all $p < n-d-1 = \dim C$. In particular, if $d = n-1$, namely if C is a point, this is automatically true.

(ii) Assume that

$$H^i(C, \mathcal{L}_Y^{\otimes -m} \otimes f_*\mathcal{O}_E(l-d-1)) = 0 \tag{13}$$

for $i > 0$ and all l with $ma + d \geq l$, where we define $\mathcal{O}_E(1)$ to be the tautological line bundle of the $\mathbb{P}^d$-bundle $E \to C$. Then $E_2^{p,d} = 0$ for all p with $p < n - d - 1$.

Proof. (i) First of all, we have

$$\begin{aligned} &\mathrm{Hom}^i_X(\mathcal{L}_X, F_{m*}\mathcal{O}_X) = \mathrm{Hom}^i_X(F_m^*\mathcal{L}_X, \mathcal{O}_X) \\ =& H^{n-i}(X, \mathcal{L}_X^{\otimes m} \otimes \omega_X)^\vee = H^{n-i}(X, \mathcal{L}_X^{\otimes m} \otimes f^*\omega_Y \otimes \mathcal{O}_X(dE))^\vee \\ =& H^{n-i}(X, f^*(\mathcal{L}_Y^{\otimes m} \otimes \omega_X) \otimes \mathcal{O}_X((ma+d)E))^\vee \\ =& (E^{n-i})^\vee. \end{aligned}$$

Hence (12) means that

$$E^{p+q} = 0 \tag{14}$$

for all $p + q \neq n$. Similarly it is easy to see that

$$(E_2^{n-i,0})^\vee = \mathrm{Hom}^i_Y(\mathcal{L}_Y, F_{m*}\mathcal{O}_Y).$$

Therefore what we have to show is that, under the assumption (14), $E_2^{p,0} = 0$ for all $p < n$ is equivalent to $E_2^{p,d} = 0$ for all $p < n - d - 1$. More strongly, we will see below $E_2^{p,d} \cong E_2^{p+d+1,0}$ for $p < n - d - 1$.

Note that

$$\mathbb{R}^q f_* \mathcal{O}_E(lE) = 0 \tag{15}$$

unless $q = 0, d$, since $f|_E \colon E \to C$ is a $\mathbb{P}^d$-bundle. We also have

$$f_* \mathcal{O}_E(lE) = 0$$

for all positive l. Then we have a short exact sequence

$$\begin{aligned} 0 \to \mathcal{L}_Y^{\otimes m} \otimes \omega_Y \otimes \mathbb{R}^q f_* \mathcal{O}_X((l-1)E) \to & \mathcal{L}_Y^{\otimes m} \otimes \omega_Y \otimes \mathbb{R}^q f_* \mathcal{O}_X(lE) \\ \to & \mathcal{L}_Y^{\otimes m} \otimes \omega_Y \otimes \mathbb{R}^q f_* \mathcal{O}_E(lE) \to 0 \end{aligned} \tag{16}$$

for $l \geq 0$ and all q. Hence by the vanishing $\mathbb{R}^q f_* \mathcal{O}_X = 0$ for $q \neq 0$ and (15), we conclude that

$$\begin{aligned} E_2^{p,q} = & H^p(Y, \mathcal{L}_Y^{\otimes m} \otimes \omega_Y \otimes \mathbb{R}^q f_* \mathcal{O}_X((ma+d)E)) \\ \cong & H^p(Y, \mathcal{L}_Y^{\otimes m} \otimes \omega_Y \otimes \mathbb{R}^q f_* \mathcal{O}_X((ma+d-1)E)) \\ \cong & \cdots \\ \cong & H^p(Y, \mathcal{L}_Y^{\otimes m} \otimes \omega_Y \otimes \mathbb{R}^q f_* \mathcal{O}_X) = 0 \end{aligned}$$

for all p and all $q \neq 0, d$. Thus we have $E_2^{p,q} \cong E_{d+1}^{p,q}$ for all p, q. Therefore from (14) we obtain

$$E_2^{p,d} \cong E_{d+1}^{p,d} \cong E_{d+1}^{p+d+1,0} \cong E_2^{p+d+1,0}$$

for $p + d + 1 < n$. Thus we obtain the conclusion.

(ii) By the duality,

$$\begin{aligned} & H^{n-d-1-p}(C, \mathcal{L}_Y^{\otimes -m} \otimes f_* \mathcal{O}_E(l-d-1))^\vee \\ = & H^p(C, \mathcal{L}_Y^{\otimes m} \otimes (f_* \mathcal{O}_E(l-d-1))^\vee \otimes \omega_C) \\ = & H^p(Y, \mathcal{L}_Y^{\otimes m} \otimes \omega_Y \otimes \mathbb{R}^d f_* \mathcal{O}_E(lE)). \end{aligned}$$

By the assumption (13), the last one vanishes for all l, p with $ma + d \geq l$ and $p < n - d - 1$. Then the vanishing of $E_2^{p,d}$ is a direct consequence of the vanishing $\mathbb{R}^d f_* \mathcal{O}_X = 0$ and (16). $\square$

Theorem 6.3. *Let X be a toric del Pezzo surface. Then $\mathfrak{D}_X$ is a full strong exceptional collection on X.*

Proof. We have already checked the statement for the maximal del Pezzo surface Y_3 in §4.1. Then the statement for the other cases follows from Lemmas 5.1 and 6.2. $\square$

See also [15, Theorem 8.2] for an interesting result in this direction.

Lemma 6.4. *In the notation in Lemma* 6.2*, assume that X and Y are smooth toric Fano* 3*-folds and that the vanishing*

$$\operatorname{Hom}_X^i(\mathcal{L}_X, F_{m*}\mathcal{O}_X) = 0 \tag{17}$$

holds for all $i > 0$. Then the vanishing

$$\operatorname{Hom}_Y^i(\mathcal{L}_Y, F_{m*}\mathcal{O}_Y) = 0$$

holds for all $i > 0$.

Proof. We divide the proof into two parts.

Step 1. By the last assertion in Lemma 6.2(i), we may assume that $C \cong \mathbb{P}^1$. There are primitive generators $\boldsymbol{v}_1, \dots, \boldsymbol{v}_5$ of 1-dimensional cones in Δ_Y (and we sometimes regard them as generators of 1-dimensional cones in Δ_X) such that

- C is the T-invariant curve corresponding to the 2-dimensional cone generated by $\boldsymbol{v}_1$ and $\boldsymbol{v}_4$, and
- the sets $\{\boldsymbol{v}_1, \boldsymbol{v}_4, \boldsymbol{v}_5\}$, $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_4\}$ and $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ generate 3-dimensional cones in Δ_Y respectively.
- $\boldsymbol{v}_3$ is different from $\boldsymbol{v}_4$, but it may coincide with $\boldsymbol{v}_5$.

We have the following equalities

$$\boldsymbol{v}_2 + \boldsymbol{v}_5 + \alpha\boldsymbol{v}_1 + \beta\boldsymbol{v}_4 = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{v}_3 + \boldsymbol{v}_4 + \gamma\boldsymbol{v}_1 + \delta\boldsymbol{v}_2 = \boldsymbol{0}.$$

for some $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$ (see Figure 6). Without the loss of generality, we may assume that $\beta \geq \alpha$. Then we know that $\beta \geq 0$, since $\alpha + \beta \geq -1$ by the condition that Y is Fano [21, Page 89]. By these equalities, we have

$$\boldsymbol{v}_5 - \beta\boldsymbol{v}_3 + (1 - \beta\delta)\boldsymbol{v}_2 + (\alpha - \beta\gamma)\boldsymbol{v}_1 = 0. \tag{18}$$

Note that the set $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ also generates a 3-dimensional cone, say σ, in Δ_X. So we can apply Lemma 6.1 for σ. Take $\boldsymbol{u} \in P_m^3$ such that $\mathcal{L}_Y \cong \mathcal{O}_X(D^Y_{\boldsymbol{u},\varphi(\sigma)})$, and denote by $D_i (= D_i^Y)$ (resp. D_i^X) the prime divisors on Y (resp. X) corresponding to $\boldsymbol{v}_i$, and we put q_i to be the coefficient of D_i in a T-invariant divisor $D^Y_{\boldsymbol{u},\varphi(\sigma)}$ on Y, namely we have

$$D^Y_{\boldsymbol{u},\varphi(\sigma)} = q_1 D_1 + q_2 D_2 + q_3 D_3 + q_4 D_4 + \cdots .$$

We have

$$D^X_{\boldsymbol{u},\sigma} = f^* D^Y_{\boldsymbol{u},\varphi(\sigma)} + aE, \quad \text{that is} \quad \mathcal{L}_X \cong f^*\mathcal{L}_Y \otimes \mathcal{O}_X(aE)$$

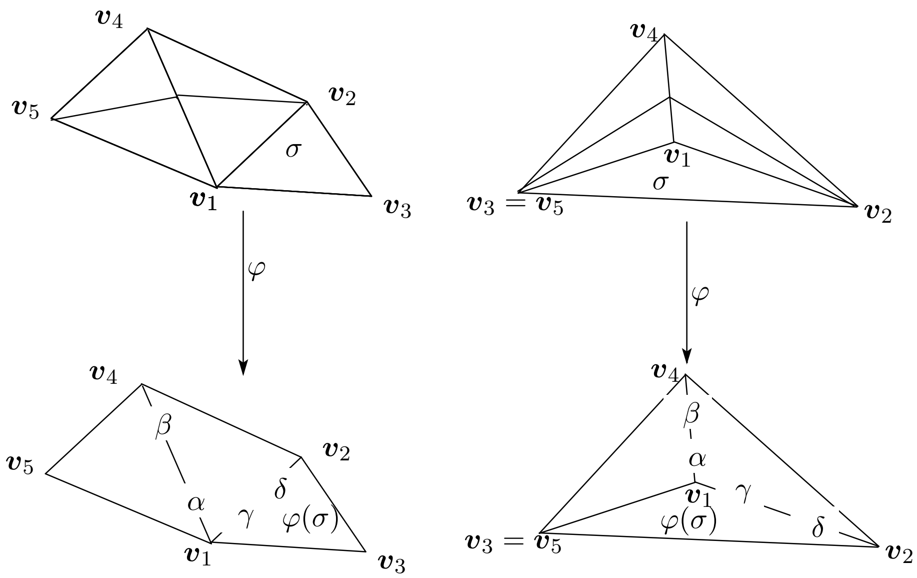


Figure 6: Divisorial contractions

for some $a \geq 0$ as in Lemma 6.1.

To check (13) in the case $C \cong \mathbb{P}^1$, it is enough to show

$$H^1(C, \mathcal{L}_Y^{\otimes -m} \otimes f_* \mathcal{O}_E(ma-1)) = 0. \tag{19}$$

By choosing $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ as a basis of the lattice $N \cong \mathbb{Z}^3$, we obtain from (18) that

$$\boldsymbol{v}_5 = {}^t(\beta\gamma - \alpha, \beta\delta - 1, \beta) \tag{20}$$

and $q_1 = q_1^m(\boldsymbol{u}, \varphi(\sigma)) = 0$, and $q_2 = q_2^m(\boldsymbol{u}, \varphi(\sigma)) = 0$. We also know by (2) that $\mathcal{N}_{C/Y} \cong \mathcal{O}_C(\alpha) \oplus \mathcal{O}_C(\beta)$. In particular, we have

$$f_* \mathcal{O}_E(i) \cong Sym^i \mathcal{N}_{C/Y}^\vee \cong \mathcal{O}_C(-i\alpha) \oplus \mathcal{O}_C(-(i-1)\alpha - \beta) \oplus \cdots \oplus \mathcal{O}_C(-i\beta)$$

for $i \geq 0$. We denote F a fiber of $\mathbb{P}^1$-bundle $f|_E \colon E \to C$. Then we note that T-invariant prime divisors on X which intersect with F are only D_1^X, D_2^X, D_4^X and D_5^X (and of course, F is contained in E). Thus we have

$$-a = aE \cdot F = (D^X_{\boldsymbol{u},\sigma} - f^* D^Y_{\boldsymbol{u},\varphi(\sigma)}) \cdot F = D^X_{\boldsymbol{u},\sigma} \cdot F = q_4,$$

since $D_2^X \cdot F = D_5^X \cdot F = 0$ and $q_1 = 0$. Combining this with

$$\deg \mathcal{L}_Y|_C = (q_4 D_4 + q_5 D_5) \cdot C = \beta q_4 + q_5,$$

we have

$$\begin{aligned}
&\deg \mathcal{L}_Y^{\otimes -m} \otimes O_C((1-ma)\beta) = -m(\beta q_4 + q_5) + (1-ma)\beta \\
=& -m(\beta q_4 + q_5) + (1+mq_4)\beta = -mq_5 + \beta.
\end{aligned}$$

Since $\beta \geq \alpha$ and $\beta \geq 0$, we know that $q_5 \leqq 0$ if and only if (19) is true for $m \gg 0$.

By (20), we have

$$q_5 = \lfloor \frac{(\beta\gamma - \alpha)x + (\beta\delta - 1)y + \beta z}{m} \rfloor$$

for $\boldsymbol{u} = {}^t(x, y, z) \in P_m^3$. By observing Figure 4, we can see that

- in all cases, we have $\beta\delta - 1 \leqq 0$,
- if $\beta \geq 2$, X is in (11) and Y is in (4) in Theorem 3.1,
- if $\beta = 1$, $\beta\gamma - \alpha \leqq 0$, and
- if $\beta \leqq 0$ (then actually $\beta = 0$), $\beta\gamma - \alpha \leqq 1$.

Consequently, we obtain $q_5 \leqq 0$, except the case X is in (11) and Y is in (4).

Step 2. Let X be the Fano 3-fold in (11) and take $\mathcal{L}_X \in \mathfrak{D}_X$. Then we have seen in §4.2 that $\mathcal{L}_X \ncong \mathcal{O}_X(-D_4 - D_5 + D_6)$ if and only if the equality (17) holds for all $i > 0$. Note that $\boldsymbol{v}_6$ in Figure 5 plays the role of $\boldsymbol{v}_5$ in Figure 6. Consequently we know that if $\mathcal{L}_X$ is not isomorphic to $\mathcal{O}_X(-D_4 - D_5 + D_6)$, then $q_5 \leq 0$ by the computation in §4.2. Then the result follows. □

Now we give the proof of Theorem 1.1.

Proof. In §5 we have already seen that $F_{m*}\mathcal{O}_X$ is a generator for maximal smooth toric Fano 3-folds X. We have also seen in §5.1 and Proposition 5.5 that $F_{m*}\mathcal{O}_X$ is a tilting generator for the Fano 3-folds in (17) and (18). Then Lemmas 5.1 and 6.4 imply that $\mathfrak{D}_X$ is a full strong exceptional collection for all smooth toric Fano 3-folds except the cases (4) and (11).

For the case X in (11), we have seen in Proposition 5.3 that the set $\mathfrak{D}_{nef}$ is a full strong exceptional collection on X, and in §4.2 that

$$\mathrm{Hom}_X^i(\mathcal{L}_X, F_{m*}\mathcal{O}_X) = 0$$

holds for all $i > 0$ and all $\mathcal{L}_X \in \mathfrak{D}_{nef}$. Take the Fano 3-fold Y in (4) and consider the blowing-up $f\colon X \to Y$ in Figure 3. Then Lemmas 5.1 and 6.4 implies that the subset $\left\{ f_*\mathcal{L}_X \;\middle|\; \mathcal{L}_X \in \mathfrak{D}_{nef} \right\}$ of $\mathfrak{D}_Y$ forms a full strong exceptional collection on Y. □

Acknowledgments

Hiroshi Sato and Yukinobu Toda are always generous with their knowledge and ideas. A part of the paper was written during my stay in Max-Planck Institute in September, 2010. I appreciate all of them for their support. I am also supported by the Grants-in-Aid for Scientific Research (No.23340011).

References

[1] V. Batyrev, Toroidal Fano 3-folds, *Math. USSR-Izv.* **19**, (1982) 13–25.

[2] A. Bayer, Semisimple quantum cohomology and blowups, *Int. Math. Res. Not.* **40**, (2004), 2069–2083.

[3] A. Bernardi, S. Tirabassi, Derived categories of toric Fano 3-Folds via the Frobenius morphism, *Matematiche (Catania)* **64**, (2009), 117–154.

[4] A. I. Bondal, Derived categories of toric varieties. *Obervolfach Reports*, **3**, (2006), 284–286.

[5] A. I. Bondal, A. Polishchuk, Homological properties of associative algebras: the method of helices, *Russian Academy of Sciences. Izv. Math.* **42**, (1994), 219–260.

[6] A. I. Bondal, M. Van den Bergh, Generators and representability of functors in commutative and noncommutative geometry, *Mosc. Math. J.* **3**, (2003), 1–36.

[7] L. Borisov, Z. Hua, On the conjecture of King for smooth toric Deligne-Mumford stacks, *Adv. Math.* **221**, (2009), 277–301.

[8] L. Costa, R.M. Miro-Roig, Frobenius splitting and derived category of toric varieties, *Illinois J. Math.* **54**, (2010), 649–669.

[9] L. Costa, R.M. Miro-Roig, Derived category of toric varieties with small Picard numbers, *Cent. Eur. J. Math.* **10**, (2012), 1280–1291.

[10] L. Costa, S. Di Rocco, R.M. Miro-Roig, Derived category of fibrations, *Math. Res. Lett.* **18**, (2011), 425432.

[11] A. Dey, M. Lason, M. Michalek, Derived category of toric varieties with Picard number three, *Matematiche (Catania)* **64**, (2009), 99–116.

[12] A. I. Efimov, Maximal lengths of exceptional collections of line bundles, arXiv:1010.3755.

[13] W. Fulton, *Introduction to toric varieties.* Annals of Mathematics Studies, 131. Princeton University Press, Princeton, NJ, 1993. xii+157 pp.

[14] L. Hille, M. Perling, A Counterexample to King's Conjecture, *Compos. Math.* **142**, (2006), 1507–1521.

[15] L. Hille, M. Perling, Exceptional sequences of invertible sheaves on rational surfaces, *Compos. Math.* **147**, (2011), 1230–1280.

[16] D. Huybrechts, *Fourier-Mukai transforms in algebraic geometry.* Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2006. viii+307 pp.

[17] A. Ishii, K. Ueda, Dimer models and exceptional collections, arXiv:0911.4529.

[18] Y. Kawamata, Derived categories of toric varieties, *Michigan Math. J.* **54**, (2006), no. 3, 517–535.

[19] A. King, Tilting bundles on some rational surfaces, preprint.

[20] M. Lason, M. Michalek, On the full, strongly exceptional collections on toric varieties with Picard number three, *Collect. Math.* **62**, (2011), 275–296

[21] T. Oda, *Convex bodies and algebraic geometry. An introduction to the theory of toric varieties.* Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 15. Springer-Verlag, Berlin, 1988. viii+212 pp.

[22] R. Ohkawa, H. Uehara, Frobenius morphisms and derived categories on 2-dimensional toric Deligne–Mumford stacks, *Adv. Math.* **244**, (2013) 241-167.

[23] A. Samokhin, Tilting bundles via the Frobenius morphism, arXiv:0904.1235v2.

[24] Jesper F. Thomsen, Frobenius direct images of line bundles on toric varieties. *J. Algebra.* **226**, (2000), 865–874.

[25] M. Van den Bergh, Three-dimensional flops and noncommutative rings. *Duke Math. J.* **122**, (2004), 423–455.

[26] K. Watanabe, M. Watanabe, The classification of Fano 3-folds with torus embeddings, *Tokyo J. Math.* **5**, (1982), 37-48.

Hokuto Uehara

Department of Mathematics and Information Sciences, Tokyo Metropolitan University, 1-1 Minamiohsawa, Hachioji-shi, Tokyo, 192-0397, Japan

e-mail address : hokuto@tmu.ac.jp